On Certain Functional Equations, II

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The present paper is a continuation of our previous paper [2]. In that paper we considered certain functional equations and proved the following fact.

Let $A(z)$, $B(z)$, $P(z)$ and $Q(z)$ be nonconstant regular functions in the annulus $0<|z|<+\infty$. Suppose that these four functions satisfy the functional equation

$$(*)\qquad P(z)-Q(z)=A(ze^{P(z)})-B(ze^{Q(z)})$$

in this annulus. Suppose further that the functions $P(z)$ and $Q(z)$ have poles at the point at infinity. Then the order of the pole of $P(z)$ is equal to that of $Q(z)$.

The object of this paper is to give a further investigation and to show the next theorem.

Theorem. *Let $A(z)$, $B(z)$, $P(z)$ and $Q(z)$ be nonconstant regular functions satisfying the functional equation (*). Suppose that the functions $P(z)$ and $Q(z)$ both have poles at the point at infinity. Then $P(z)$ and $Q(z)$ differ by a constant.*

By changing z to $1/z$ in the functional equation (*) and by setting $A_*(z)=-A(1/z)$, $B_*(z)=-B(1/z)$, $P_*(z)=-P(1/z)$ and $Q_*(z)=-Q(1/z)$, we can easily restate our theorem as follows.

Let $A(z)$, $B(z)$, $P(z)$ and $Q(z)$ be nonconstant regular functions which satisfy the functional equation (*). If the functions $P(z)$ and $Q(z)$ have poles at the origin, then the difference $P(z)-Q(z)$ reduces to a constant function.

Let $f(z)$ be an arbitrary nonconstant regular function in the annulus $0<|z|<+\infty$, and let us set

$$S(z)=z\exp(f(z)),\quad S_0(z)=z,$$

$$S_n(z) = S_{n-1}(S(z)) \quad (n=1, 2, 3, \ldots).$$

Then it is clear by induction that

$$S_n(z) = z \exp\left(\sum_{j=0}^{n-1} f(S_j(z))\right)$$

for each natural number n. Hence if

$$A(z) = -\sum_{j=0}^{m} f(S_j(z)), \quad P(z) = \sum_{j=0}^{n} f(S_j(z)),$$

where m and n are nonnegative integers, then

$$P(z) - A(ze^{P(z)}) = \sum_{j=0}^{m+n+1} f(S_j(z)).$$

Accordingly there are various kind of functions which satisfy the functional equation (*). Furthermore our theorem fails to hold if the condition on the functions $P(z)$ and $Q(z)$ is removed.

1. **Preliminary.** Let $A(z)$, $B(z)$, $P(z)$ and $Q(z)$ be nonconstant regular functions in the annulus $0<|z|<+\infty$ which satisfy the functional equation (*). For the sake of simplicity we set

$$(1.1) \qquad P_*(z) = ze^{P(z)}, \quad Q_*(z) = ze^{Q(z)}.$$

Then the functional equation (*) becomes

$$(1.2) \qquad P(z) - Q(z) = A(P_*(z)) - B(Q_*(z)).$$

Suppose that the functions $P(z)$ and $Q(z)$ have poles at the point at infinity. Then by our previous theorem the order of the pole of $P(z)$ must be equal to that of $Q(z)$. Hence with some natural number n,

$$(1.3) \qquad P(z) = (a+o(1))z^n, \quad Q(z) = (b+o(1))z^n$$

near the point at infinity, where a and b are nonzero constants. Hereafter we may suppose without loss of generality that the constant b is real positive. For the function $Q(z)$, by making use of (1.3), we can define $2n$ differentiable functions $z_j(t)$ $(j=1, \dots, 2n)$ of t for $t \geqq 0$ such that

$$\lim_{t\to\infty} |z_j(t)| = +\infty, \tag{1.4}$$

$$\lim_{t\to\infty} \arg z_j(t) = (2j-1)\pi/2n, \tag{1.5}$$

$$Q_*(z_j(t)) = R\exp((-1)^{j+1} it) \tag{1.6}$$

for $t \geqq 0$, where R is a suitable real positive constant. With the help of (1.4)-(1.6) we easily deduce

$$\lim_{t\to\infty} \frac{t}{(z_j(t))^n} = (-1)^j ib. \tag{1.7}$$

We set the functions

$$w_j(t) = P_*(z_j(t)) \tag{1.8}$$

for $t \geqq 0$ $(j=1, \dots, 2n)$. Then the functional equation (1.2) yields

$$P(z_j(t)) - Q(z_j(t)) = A(w_j(t)) - B(Q_*(z_j(t))),$$

so that by means of (1.3), (1.4) and (1.6),

$$\lim_{t\to\infty} \frac{A(w_j(t))}{(z_j(t))^n} = a-b \tag{1.9}$$

for $j=1, \dots, 2n$. On the other hand by the definitions (1.1) and (1.8), and by the property (1.6),

$$\frac{w_j'(t)}{w_j(t)} = (-1)^{j+1} i \frac{1+z_j(t)P'(z_j(t))}{1+z_j(t)Q'(z_j(t))}, \tag{1.10}$$

so that the asymptotic behaviours (1.3) imply

$$\lim_{t\to\infty} \frac{w_j'(t)}{w_j(t)} = (-1)^{j+1} i \frac{a}{b} \tag{1.11}$$

for $j=1, \ldots, 2n$. Our task is to analyze these logarithmic derivatives $w_j'(t)/w_j(t)$ explicitly. For this purpose we introduce the auxiliary function

$$f(z) = Q(z) - \frac{b}{a} P(z). \tag{1.12}$$

This function $f(z)$ is analytic in the annulus $0 < |z| < +\infty$, and it is regular or has a pole at the point at infinity. In the latter case its order of the pole is at most $n-1$.

In what follows our consideration is divided into the following five cases.

Case 1 Neither the real part nor the imaginary part of the constant a is zero.

Case 2 The constant a is purely imaginary and the function $f(z)$ is regular at the point at infinity.

Case 3 The constant a is purely imaginary and the function $f(z)$ has a pole at the point at infinity.

Case 4 The constant a is real and the function $f(z)$ is regular at the point at infinity.

Case 5 The constant a is real and the function $f(z)$ has a pole at the point at infinity.

2. **Construction of curves.** First of all we consider the case 1. From now on we denote the real part and the imaginary part of the constant a by a^* and a_*, respectively.

Let u be a positive number such that

$$|a_*| \log \frac{R+u}{R} > 4b\pi, \tag{2.1}$$

where R is the real positive quantity of (1.6). Let l be the real positive number defined with

$$l = \exp\left(\frac{3}{2nb} \log \frac{R+u}{R}\right). \tag{2.2}$$

It is clear from (1.3) that

$$1 + zP'(z) = (na + o(1))z^n,$$

$$1 + zQ'(z) = (nb + o(1))z^n$$

near the point at infinity. Hence it is possible to find a positive number R_* (>1) such that

$$\left|\frac{z^n}{1+zQ'(z)} - \frac{1}{nb}\right| \leqq \frac{1}{2nb}, \tag{2.3}$$

$$\left|\frac{1+zP'(z)}{1+zQ'(z)} - \frac{a}{b}\right| \leqq \frac{1}{2b}\min(|a^*|, |a_*|) \tag{2.4}$$

for values of z with $|z| \geqq R_*$. Since $1+zQ'(z)$ is not identically zero by assumption, there surely exists a real number γ $(|\gamma| \leqq \pi)$ such that all the roots of the equation $\arg Q_*(z) = \gamma$ satisfy $Q_*'(z) \neq 0$. Let k be an even or odd integer between 1 and $2n$ according to whether the product a^*a_* is positive or negative, and we set

$$t_j = (-1)^{k+1}\gamma + 2\pi j \qquad (j = 1, 2, \ldots). \tag{2.5}$$

Under these notations we prove the following

Lemma 1. *If* $|z_k(t_j)| > lR_*$, *then there exists the differentiable function* $u_j(t)$ *of* t *for the interval* $0 \leqq t \leqq u$ *such that* $u_j(0) = z_k(t_j)$ *and*

$$Q_*(u_j(t)) = (R+t)e^{i\gamma} \tag{2.6}$$

for $0 \leqq t \leqq u$. *Furthermore the double inequality*

$$\frac{1}{l} \leqq \left|\frac{u_j(t)}{u_j(0)}\right| \leqq l \tag{2.7}$$

holds there.

Proof. It is clear from (1.6) and the definition (2.5) that $Q_*(z_k(t_j)) =$

$Re^{i\gamma}$, so that $Q'_*(z_k(t_j)) \neq 0$. Hence it is possible to define a differentiable function $u_j(t)$ of t for $0 \leqq t \leqq u'$ satisfying $u_j(0) = z_k(t_j)$ and the above (2.6). Here u' is a real positive number and it may be very small in general. Since l is greater than one and $|u_j(0)| > lR_*$ by assumption, we may suppose from the continuity that $|u_j(t)| > R_*$ for $0 \leqq t \leqq u'$. It follows from (2.6) that

$$(R+t)Q'_*(u_j(t))u'_j(t) = Q_*(u_j(t))$$

for $0 \leqq t \leqq u'$. Hereby from the definition (1.1),

$$(R+t)\frac{u'_j(t)}{u_j(t)} = \frac{1}{1+u_j(t)Q'(u_j(t))}$$

there. Since $|u_j(t)| > R_* > 1$, the inequality (2.3) yields

$$(R+t)\left|\frac{u'_j(t)}{u_j(t)}\right| \leqq \frac{3}{2nb}$$

for $0 \leqq t \leqq u'$. It therefore follows that

$$\left|\log\left|\frac{u_j(t)}{u_j(0)}\right|\right| \leqq \frac{3}{2nb}\log\frac{R+t}{R}$$

for $0 \leqq t \leqq u'$. Hence if $u' \leqq u$, then the double inequality (2.7) holds for $0 \leqq t \leqq u'$, because of the definition (2.2). By this fact we can extend the function $u_j(t)$ analytically to the function defined on the interval $0 \leqq t \leqq u$. Of course this extension satisfies (2.6) and (2.7). This completes the proof.

Evidently from the definition (2.5), the numbers t_j go to infinity with j, so that by means of (1.4), $|z_k(t_j)| > lR_*$ for all large j. Hence we can define the functions $u_j(t)$ on the interval $0 \leqq t \leqq u$ which satisfy (2.6) and (2.7). We now set

$$v_j(t) = P_*(u_j(t)) \tag{2.8}$$

for $0 \leqq t \leqq u$. It then follows from (2.6) that

$$(2.9)\qquad (R+t)\frac{v_j'(t)}{v_j(t)}=\frac{1+u_j(t)P'(u_j(t))}{1+u_j(t)Q'(u_j(t))}$$

for $0 \leqq t \leqq u$. Since $|u_j(t)| > R_*$ by (2.7), the inequality (2.4) implies

$$(2.10)\qquad \left|(R+t)\frac{v_j'(t)}{v_j(t)}-\frac{a}{b}\right| \leqq \frac{1}{2b}\min(|a^*|, |a_*|)$$

for real values of t with $0 \leqq t \leqq u$. On the other hand we insert the functions $u_j(t)$ into the functional equation (1.2). Then we deduce

$$P(u_j(t)) - Q(u_j(t)) = A(v_j(t)) - B(Q_*(u_j(t)))$$

for $0 \leqq t \leqq u$. It thus follows from (2.6) that the functions

$$A(v_j(t)) - P(u_j(t)) + Q(u_j(t))$$

are bounded uniformly. Furthermore with the aid of (2.7), the functions $u_j(t)$ become infinite uniformly as j tends to infinity. It therefore follows from (1.3) that

$$(2.11)\qquad \lim_{j\to\infty}\frac{A(v_j(t))}{(u_j(t))^n}=a-b$$

uniformly for $0 \leqq t \leqq u$.

2.1. In this subsection we assume that the real part of the constant a is positive, that is, that $a^* > 0$. Let t^* be a positive number such that $|z_k(t)| > R_*$ for $t \geqq t^*$. By referring to (1.10) and (2.4), we have

$$(2.12)\qquad \left|(-1)^k i\frac{w_k'(t)}{w_k(t)}-\frac{a}{b}\right| \leqq \frac{1}{2b}\min(|a^*|, |a_*|)$$

for all values of t with $t \geqq t^*$. Since $(-1)^k a^* a_* > 0$ by definition and $a^* > 0$ by assumption, $(-1)^k a_*$ must be positive. Hence the above (2.12) gives

$$(2.13)\qquad \mathrm{Re}\,\frac{w_k'(t)}{w_k(t)} \geqq \frac{|a_*|}{2b},$$

$$(2.14) \qquad (-1)^{k+1}\operatorname{Im}\frac{w_k'(t)}{w_k(t)} \geqq \frac{a^*}{2b}$$

for $t \geqq t^*$. By virtue of (2.13), the function $|w_k(t)|$ is strictly increasing for $t \geqq t^*$ and becomes infinite when t goes to infinity. More precisely by means of (1.11), using l'Hospital's rule, we have

$$\lim_{t\to\infty}\frac{1}{t}\log|w_k(t)| = \frac{|a_*|}{b}.$$

It therefore follows from (1.7) that

$$(2.15) \qquad \lim_{t\to\infty}\frac{(z_k(t))^n}{w_k(t)} = 0,$$

so that by means of (1.9),

$$(2.16) \qquad \lim_{t\to\infty}\frac{A(w_k(t))}{w_k(t)} = 0.$$

By $p(x)$ we denote the inverse function of $|w_k(t)|$ defined for $x \geqq x^*$, where $x^* = |w_k(t^*)|$. Then this function $p(x)$ satisfies $p(x^*) = t^*$ and

$$(2.17) \qquad |w_k(p(x))| = x$$

for $x \geqq x^*$. Let $I(t)$ stand for a branch of the argument of $w_k(t)$ for $t \geqq t^*$. By means of (2.14), $(-1)^{k+1}I(t)$ is increasing steadily there, so that the composite $(-1)^{k+1}I(p(x))$ also increases for $x \geqq x^*$. Furthermore by definition,

$$(2.18) \qquad w_k(p(x)) = x\ \exp(iI(p(x)))$$

for real values of x with $x \geqq x^*$.

Now let j be an arbitrary large integer. Then we may suppose that the number t_j defined with (2.5) satisfies $t_j \geqq t^*$ and $|z_k(t_j)| > lR_*$. For this number t_j we have defined the functions $u_j(t)$ and $v_j(t)$ on the interval $0 \leqq t \leqq u$. Since $u_j(0) = z_k(t_j)$,

$$v_j(0) = P_*(u_j(0)) = P_*(z_k(t_j)) = w_k(t_j) \tag{2.19}$$

from (1.8) and (2.8). The function $v_j(t)$ satisfies (2.10). It thus follows that

$$(R+t)\ \mathrm{Re}\,\frac{v_j'(t)}{v_j(t)} \geqq \frac{a^*}{2b}, \tag{2.20}$$

$$(-1)^k(R+t)\ \mathrm{Im}\,\frac{v_j'(t)}{v_j(t)} \geqq \frac{|a_*|}{2b} \tag{2.21}$$

for $0 \leqq t \leqq u$. Hence $|v_j(t)|$ increases steadily there. For convenience we denote $|v_j(0)|$ and $|v_j(u)|$ by x_j and x_j', respectively. Then the inverse of $|v_j(t)|$ can be defined on the interval $x_j \leqq x \leqq x_j'$ and moves increasingly from 0 to u when x does from x_j to x_j'. With $q(x)$ we denote this inverse function. Note that

$$x_j = |v_j(0)| \geqq |w_k(t^*)| = x^*$$

by (2.19) and $t_j \geqq t^*$. We next set

$$J(t) = \arg\ v_j(t)$$

for $0 \leqq t \leqq u$, where the branch is chosen so that $J(0) = I(t_j)$. Then with the help of (2.21) this argument $J(t)$ increases or decreases steadily for $0 \leqq t \leqq u$ according to whether k is even or odd. In particular the composite function $(-1)^kJ(q(x))$ is a strictly increasing function of x for $x_j \leqq x \leqq x_j'$. Furthermore by means of (2.1) and (2.21) again,

$$(-1)^k(J(u) - J(0)) \geqq \frac{|a_*|}{2b}\log\frac{R+u}{R} > 2\pi. \tag{2.22}$$

In addition to these facts it is clearly true by the definitions of $q(x)$ and $J(t)$ that

$$v_j(q(x)) = x\ \exp(iJ(q(x))) \tag{2.23}$$

for real values of x with $x_j \leqq x \leqq x_j'$. Consider the function

$$L(x) = (-1)^{k+1} I(p(x)) + (-1)^k J(q(x))$$

on the interval $x_j \leqq x \leqq x_j'$. Then $L(x)$ is a continuous strictly increasing function of x. Since $p(x_j) = t_j$, $q(x_j) = 0$ and $I(t_j) = J(0)$, we have $L(x_j) = 0$. On the other hand $L(x_j') > 2\pi$ from (2.22). Hereby we can find a point x_j'' of the interval such that $L(x_j'') = 2\pi$. To simplify the notation we set $t_j'' = p(x_j'')$ and $u_j = q(x_j'')$. It then follows that

$$J(u_j) = (-1)^k 2\pi + I(t_j''). \tag{2.24}$$

We now define the arcs

$$\begin{aligned} C_j^+ &= \{w_k(t) : t_j \leqq t \leqq t_j''\}, \\ C_j^- &= \{v_j(t) : 0 \leqq t \leqq u_j\}. \end{aligned} \tag{2.25}$$

With the help of (2.18) and (2.23), these two arcs are simple and lie entirely in the annulus $x_j \leqq |z| \leqq x_j''$. Furthermore by means of (2.18), (2.23) and (2.24), we have $w_k(t_j'') = v_j(u_j)$. Hence the arcs C_j^+ and C_j^- have the same initial point and the same final point. Here we divide the annulus $x_j \leqq |z| \leqq x_j''$ in two parts by the rays $\arg z = I(t_j)$, $I(t_j'')$. Then by what is shown above the arc C_j^+ lies entirely in one part while the arc C_j^- does in the other. Accordingly the curve $C_j = C_j^+ - C_j^-$ which consists of the arc C_j^+ and the arc C_j^- in the opposite direction must be simple closed and lie entirely in the annulus $|w_k(t_j)| \leqq |z| \leqq |w_k(t_j'')|$. Evidently the winding number of C_j with respect to the origin is 1 or -1.

For all sufficiently large integers j we have defined the functions $u_j(t)$ and $v_j(t)$ on the interval $0 \leqq t \leqq u$, and using the functions $v_j(t)$, we have constructed the simple closed curves C_j. We need one more piece of information. Since $|v_j(t)|$ are increasing for $0 \leqq t \leqq u$ and $v_j(0) = w_k(t_j)$ by (2.19),

$$|v_j(t)| \geqq |w_k(t_j)|$$

for $0 \leqq t \leqq u$. Hence with the aid of (2.7) again,

$$\frac{|u_j(t)|^n}{|v_j(t)|} \leqq \frac{l^n |z_k(t_j)|^n}{|w_k(t_j)|}$$

there. It thus follows from (2.15) that

$$\lim_{j\to\infty} \frac{(u_j(t))^n}{v_j(t)} = 0 \tag{2.26}$$

uniformly for $0 \leqq t \leqq u$. Combining this (2.26) with (2.11) we finally obtain

$$\lim_{j\to\infty} \frac{A(v_j(t))}{v_j(t)} = 0 \tag{2.27}$$

uniformly for $0 \leqq t \leqq u$. It therefore follows from (2.16) and this (2.27) that the function $A(z)/z$ converges to zero when z approaches the point at infinity through the sequence of the simple closed curves C_j. Accordingly by the maximum modulus principle and by the nature of the curves C_j, this function $A(z)/z$ converges to zero as z tends to infinity. Hence $A(z)/z$ is regular and takes the value zero at the point at infinity. Consequently the function $A(z)$ must be regular at the point at infinity. However this clearly contradicts (1.9).

2.2. Next we suppose that the real part of the constant a is negative, that is, that $a^* < 0$. Then $(-1)^k a_*$ is also negative and the estimate (2.12) yields

$$\operatorname{Re} \frac{w_k'(t)}{w_k(t)} \leqq -\frac{|a_*|}{2b},$$

$$(-1)^{k+1} \operatorname{Im} \frac{w_k'(t)}{w_k(t)} \leqq \frac{a^*}{2b} \tag{2.28}$$

for $t \geqq t^*$, where t^* is the real quantity defined in the previous subsection. This time the function $|w_k(t)|$ is strictly decreasing for $t \geqq t^*$ and converges to zero as t becomes infinite. Further by using (1.11) we obtain

$$\lim_{t\to\infty} \frac{1}{t} \log |w_k(t)| = -\frac{|a_*|}{b} < 0 . \tag{2.29}$$

It thus follows from (1.7) and (2.29) that

$$\lim_{t\to\infty} (z_k(t))^n w_k(t) = 0, \tag{2.30}$$

so that

$$\lim_{t\to\infty} w_k(t) A(w_k(t)) = 0 \tag{2.31}$$

by means of (1.9). As before we denote the inverse of $|w_k(t)|$ by $p(x)$. Then $p(x)$ is defined for $0 < x \leqq x^*$ and increases from t^* to infinite when x decreases from x^* to 0, where $x^* = |w_k(t^*)|$. Let $I(t)$ stand for a branch of the argument of $w_k(t)$ for $t \geqq t^*$. With the aid of (2.28), $(-1)^k I(t)$ is an increasing function of t for $t \geqq t^*$. Hence the composite $(-1)^k I(p(x))$ increases steadily when x decreases from x^* to 0. We next consider the function $v_j(t)$ defined with (2.8). It then follows from (2.10) that

$$(R+t)\,\mathrm{Re}\,\frac{v_j'(t)}{v_j(t)} \leqq \frac{a^*}{2b},$$

$$(-1)^{k+1}(R+t)\,\mathrm{Im}\,\frac{v_j'(t)}{v_j(t)} \geqq \frac{|a_*|}{2b} \tag{2.32}$$

for $0 \leqq t \leqq u$. Hence $|v_j(t)|$ decreases monotonically there. Setting $x_j = |v_j(0)|$ and $x_j' = |v_j(u)|$, we can define the inverse of $|v_j(t)|$ on the interval $x_j' \leqq x \leqq x_j$. We denote this inverse function by $q(x)$. Let $J(t)$ stand for the branch of the argument of $v_j(t)$ with $J(0) = I(t_j)$. Then by means of (2.32), $(-1)^{k+1}J(t)$ increases monotonically and satisfies

$$(-1)^{k+1}(J(u) - J(0)) \geqq \frac{|a_*|}{2b} \log \frac{R+u}{R} > 2\pi . \tag{2.33}$$

Note that $x_j = |v_j(0)| = |w_k(t_j)| \leqq |w_k(t^*)| = x^*$. Then the function

$$L(x) = (-1)^k I(p(x)) + (-1)^{k+1} J(q(x))$$

for $x_j' \leqq x \leqq x_j$ is continuous strictly decreasing, $L(x_j) = 0$ and $L(x_j') > 2\pi$ by means of (2.33). We can thus take a point x_j'' such that $L(x_j'') = 2\pi$ and $x_j' < x_j'' < x_j$. We put $t_j'' = p(x_j'')$ and $u_j = q(x_j'')$. Then $t_j < t_j''$ and $0 < u_j < u$. With these notations we can define the simple arcs C_j^+ and C_j^- by (2.25), and hence the simple closed curve C_j winding around the origin. This time the curve C_j lies entirely in the annulus $|w_k(t_j'')| \leqq |z| \leqq |w_k(t_j)|$. Therefore the sequence of the curves C_j converges to the origin when j goes to infinity. Taking (2.7) and (2.19) into account we deduce

$$|(u_j(t))^n v_j(t)| \leqq l^n |(z_k(t_j))^n w_k(t_j)|$$

for $0 \leqq t \leqq u$. It thus follows from (2.30) that

$$\lim_{j\to\infty} (u_j(t))^n v_j(t) = 0, \tag{2.34}$$

so that with the aid of (2.11)

$$\lim_{j\to\infty} v_j(t) A(v_j(t)) = 0 \tag{2.35}$$

uniformly for $0 \leqq t \leqq u$.

Consequently by the definition of the curves C_j and by means of (2.31) and (2.35), the function $zA(z)$ converges to zero as z approaches the origin through the sequence of the curves C_j. It thus follows from the maximum modulus principle that $zA(z)$ is regular at the origin and it takes the value zero there. Hence $A(z)$ is also regular at the origin. However this contradicts (1.9). Accordingly the case 1 never occurs.

3. **Case 2.** We suppose that the constant a is purely imaginary and the auxiliary function $f(z)$ is regular at the point at infinity. Let k be an integer between 1 and $2n$ satisfying $(-1)^k a_* > 0$, where a_* is the imaginary part of a, that is, $a = ia_*$. It then follows from (1.11) that

$$\lim_{t\to\infty} \operatorname{Re} \frac{w_k'(t)}{w_k(t)} = (-1)^k \frac{a_*}{b} > 0. \tag{3.1}$$

Hence (2.13) is true for all sufficiently large t. Hereby (2.15) holds, so does (2.16).

For the imaginary part of $w_k'(t)/w_k(t)$ we need a precise estimate. For the sake of simplicity we set

$$F(z)=\frac{1+zP'(z)}{1+zQ'(z)}. \tag{3.2}$$

Then by making use of the auxiliary function $f(z)$, we deduce

$$F(z)-\frac{a}{b}=\frac{b-a-azf'(z)}{b(1+zQ'(z))}.$$

Since $f(z)$ is regular at the point at infinity, $z^2f'(z)$ is bounded there. It therefore follows from (1.3) that

$$\lim_{z\to\infty}\left(F(z)-\frac{a}{b}\right)z^n=\frac{b-a}{nb^2}. \tag{3.3}$$

Lemma 2.

$$\lim_{t\to\infty} t\ \mathrm{Im}\frac{w_k'(t)}{w_k(t)}=-\frac{a_*}{nb}. \tag{3.4}$$

Proof. Since $z_k(t)$ becomes infinite as t grows to infinity, the above (3.3) yields

$$\lim_{t\to\infty}\left(F(z_k(t))-\frac{a}{b}\right)(z_k(t))^n=\frac{b-a}{nb^2}.$$

Combining this with (1.7) we at once obtain

$$\lim_{t\to\infty} t\left(F(z_k(t))-\frac{a}{b}\right)=(-1)^k i\,\frac{b-a}{nb}.$$

Taking real part we have

$$\lim_{t\to\infty} t\ \mathrm{Re}(F(z_k(t)))=(-1)^k\frac{a_*}{nb}. \tag{3.5}$$

On the other hand it is clear from (1.10) and (3.2) that

$$F(z_k(t)) = (-1)^k i \frac{w_k'(t)}{w_k(t)}. \tag{3.6}$$

The desired (3.4) follows at once from (3.5) and (3.6).

Lemma 3. *There exist positive numbers R' and δ such that*

$$|z|^n \operatorname{Re} \frac{1+zP'(z)}{1+zQ'(z)} \geqq \frac{|a|}{2nb^2}$$

for $|z| \geqq R'$ and $|\arg z - s_k| \leqq \delta$, where $s_k = (2k-1)\pi/2n$.

Proof. With the help of (3.3) it is possible to take a positive number R' such that

$$|z|^n \left| F(z) - \frac{a}{b} - \frac{b-a}{nb^2 z^n} \right| \leqq \frac{|a|}{4nb^2}$$

for all values of z with $|z| \geqq R'$. Taking real part we thus have

$$\left| |z|^n \operatorname{Re}(F(z)) - \operatorname{Re}\left(\frac{b-a}{nb^2} \exp(-inz^*) \right) \right| \leqq \frac{|a|}{4nb^2} \tag{3.7}$$

for $|z| \geqq R'$, where $z^* = \arg z$. Note that $\exp(-ins_k) = (-1)^k i$ and $|a| = (-1)^k a_*$. Then it follows that

$$\operatorname{Re}\left(\frac{b-a}{nb^2} \exp(-ins_k) \right) = \frac{|a|}{nb^2}.$$

Hence we can find a positive number δ such that

$$\operatorname{Re}\left(\frac{b-a}{nb^2} \exp(-inx) \right) \geqq \frac{3|a|}{4nb^2} \tag{3.8}$$

for real values of x with $|x - s_k| \leqq \delta$. It therefore follows from (3.7) and (3.8) that

$$|z|^n \operatorname{Re}(F(z)) \geqq \frac{|a|}{2nb^2}$$

in the range $|z| \geqq R'$ and $|\arg z - s_k| \leqq \delta$. Lemma 3 is now proved.

Since the imaginary part a_* is different from zero, the real positive quantities u and l defined with (2.1) and (2.2), respectively, still have their meanings in this case 2. Let R^* be a positive number such that only (2.3) holds for all values of z with $|z| \geqq R^*$. Then we can easily see that Lemma 1 with R^* instead of R_* remains valid in the case 2.

Let j be an arbitrary positive integer such that $|z_k(t_j)| > lR^*$, $|z_k(t_j)| > lR'$ and

$$\text{(3.9)} \qquad \begin{gathered} \delta\, |z_k(t_j)|^n \geqq 2l^n \log l, \\ |\arg z_k(t_j) - s_k| \leqq \delta/2. \end{gathered}$$

It is clear from (1.4) and (1.5) that all sufficiently large integers surely satisfy the above conditions. Let $u_j(t)$ be the function of Lemma 1, and let $v_j(t)$ be the function defined by (2.8). Since $u_j(0) = z_k(t_j)$, the double inequality (2.7) is rewritten as

$$\text{(3.10)} \qquad \frac{1}{l} \leqq \left|\frac{u_j(t)}{z_k(t_j)}\right| \leqq l$$

for $0 \leqq t \leqq u$. In particular $|u_j(t)| \geqq R^*$ and $|u_j(t)| \geqq R'$ there. We next differentiate (2.6). Then we have

$$(R+t)\frac{u_j'(t)}{u_j(t)} = \frac{1}{1+u_j(t)Q'(u_j(t))}$$

for $0 \leqq t \leqq u$. It therefore follows from (2.3) that

$$(R+t)|u_j(t)|^n \left|\frac{u_j'(t)}{u_j(t)}\right| \leqq \frac{3}{2nb},$$

so that by means of (3.9) and (3.10),

$$(R+t)\left|\frac{u_j'(t)}{u_j(t)}\right| \leqq \frac{3\ l^n}{2nb\,|z_k(t_j)|^n}$$

for real values of t with $0 \leqq t \leqq u$. Hereby

$$|\arg u_j(t) - \arg u_j(0)| \leqq \frac{l^n}{|z_k(t_j)|^n} \log l$$

for $0 \leqq t \leqq u$. Combining this with (3.9) we at once deduce

$$|\arg u_j(t) - s_k| \leqq \delta$$

there. Consequently by virtue of (2.9) and Lemma 3, the real part of $v_j'(t)/v_j(t)$ is always positive in the interval $0 \leqq t \leqq u$. In particular, $|v_j(t)|$ increases steadily, so that $|v_j(t)| \geqq |w_k(t_j)|$ by (2.19). On the other hand it is clear from (1.3) that

$$(-1)^k \operatorname{Im} \frac{1+zP'(z)}{1+zQ'(z)}$$

converges to $|a|/b$ as z tends to infinity. Hence we may assume from (1.4), (2.9) and (3.10) that the inequality (2.21) still holds for $0 \leqq t \leqq u$. Hereby $(-1)^k \arg v_j(t)$ is a monotonic increasing function of t and satisfies

$$(-1)^k(\arg v_j(u) - \arg v_j(0)) > 2\pi$$

by means of (2.1) and (2.21). We now recall Lemma 2. Since $(-1)^k a_* = |a|$, $(-1)^k \operatorname{Im}(w_k'(t)/w_k(t))$ is negative for all sufficiently large t. Hence the function $(-1)^k \arg w_k(t)$ is finally decreasing.

Henceforth by exactly the same argument as in the previous subsection 2.1, we finally conclude that the function $A(z)$ is regular at the point at infinity. We thus arrive at a contradiction again. Accordingly the case 2 does not occur either.

4. **The logarithmic derivatives.** In the previous section we introduced the function $F(z)$ to estimate the logarithmic derivatives $w_j'(t)/w_j(t)$. In fact the function $F(z)$ defined with (3.2) satisfies (3.6).

Hence taking real and imaginary parts we have

$$\text{(4.1)} \qquad \operatorname{Re}(F(z_j(t))) = (-1)^{j+1} \operatorname{Im} \frac{w_j'(t)}{w_j(t)},$$

$$\text{(4.2)} \qquad \operatorname{Im}(F(z_j(t))) = (-1)^{j} \operatorname{Re} \frac{w_j'(t)}{w_j(t)}$$

for $j=1, 2, \ldots, 2n$. Let $f(z)$ be the auxiliary function defined by (1.12). Then we have already shown

$$\text{(4.3)} \qquad F(z) - \frac{a}{b} = \frac{b-a-azf'(z)}{b(1+zQ'(z))}$$

Assume for a moment that the function $f(z)$ has a pole of order q at the point at infinity. Then

$$\text{(4.4)} \qquad f(z) = (c+o(1))z^q$$

near the point at infinity, where c denotes a nonzero constant. Here we remark that $1 \leqq q < n$, as already mentioned. Since

$$zf'(z) = (qc+o(1))z^q$$

near the point at infinity, the above (4.3) yields

$$F(z) - \frac{a}{b} = \frac{(-qac+o(1))z^q}{(nb^2+o(1))z^n}$$

there. It hereby follows that

$$\text{(4.5)} \qquad \lim_{z\to\infty} \left(F(z) - \frac{a}{b}\right) z^{n-q} = -\frac{qac}{nb^2}.$$

On the other hand with the aid of (1.4) and (1.5), we deduce

$$\text{(4.6)} \qquad \lim_{t\to\infty} \frac{z_j(t)}{|z_j(t)|} = \exp(is_j)$$

for $j=1, 2, \ldots, 2n$, where $s_j = (2j-1)\pi/2n$. Combining this with (4.5) we thus have

$$\lim_{t\to\infty}\left(F(z_j(t))-\frac{a}{b}\right)|z_j(t)|^{n-q} = (-1)^{j+1}\, i\,\frac{qac}{nb^2}\exp(iqs_j) \tag{4.7}$$

for $j=1, 2, \ldots, 2n$.

For the case where the auxiliary function $f(z)$ is regular at the point at infinity, we have deduced (3.3) from (4.3). Therefore taking (4.6) into account we obtain

$$\lim_{t\to\infty}\left(F(z_j(t))-\frac{a}{b}\right)|z_j(t)|^{n} = (-1)^{j}\, i\,\frac{b-a}{nb^2} \tag{4.8}$$

for $j=1, 2, \ldots, 2n$.

The next three lemmas are immediate consequences of the estimate (1.11) and these (4.7) and (4.8). The first is a preliminary for the case 3.

Lemma 4. *Suppose that the constant a is purely imaginary and the function $f(z)$ has the form (4.4). Then the logarithmic derivatives $w_j'(t)/w_j(t)$ satisfy*

$$\lim_{t\to\infty}\operatorname{Re}\frac{w_j'(t)}{w_j(t)} = (-1)^j\frac{a_*}{b}, \tag{4.9}$$

$$\lim_{t\to\infty}|z_j(t)|^{n-q}\operatorname{Im}\frac{w_j'(t)}{w_j(t)} = -\frac{qa_*c^*}{nb^2}\cos(qs_j+c_*) \tag{4.10}$$

for $j=1, 2, \ldots, 2n$, where a_* is the imaginary part of a, $c^*=|c|$ and $c_*=\arg c$.

Proof. Since $a=ia_*$, the former (4.9) is an immediate consequence of (1.11). The latter (4.10) follows from (4.1) and (4.7).

Lemma 5. *Suppose that the constant a is real and the function $f(z)$ is regular at the point at infinity. Then*

$$\lim_{t\to\infty}|z_j(t)|^{n}\operatorname{Re}\frac{w_j'(t)}{w_j(t)} = \frac{b-a}{nb^2}, \tag{4.11}$$

(4.12) $$\lim_{t\to\infty} \operatorname{Im} \frac{w_j'(t)}{w_j(t)} = (-1)^{j+1} \frac{a}{b}$$

for $j=1, 2, \ldots, 2n$.

Proof. The first (4.11) follows from (4.2) and (4.8). The second (4.12) follows directly from (1.11).

Lemma 6. *If the constant* a *is real and the function* $f(z)$ *has the form* (4.4), *then*

(4.13) $$\lim_{t\to\infty} |z_j(t)|^{n-q} \operatorname{Re} \frac{w_j'(t)}{w_j(t)} = -\frac{qac^*}{nb^2} \cos(qs_j+c_*)$$

and the above (4.12) *holds for* $j=1, 2, \ldots, 2n$, *where* $c^* = |c|$ *and* $c_* = \arg c$.

Proof. The estimate (4.13) follows from (4.2) and (4.7).

Finally we want to remark some elementary facts on the values $\cos(qs_j+c_*)$. Assume that $n \geqq 2$ and that $\cos(qs_j+c_*) = 0$ for all odd integers j from 1 to $2n$. Then by the definition of s_j and by the fact $1 \leqq q < n$, we can easily see that $2q = n$ and $\sin(c_* - \pi/4) = 0$. Similarly if $\cos(qs_j+c_*) = 0$ for all even integers j between 1 and $2n$, then $2q = n$ and $\cos(c_* - \pi/4) = 0$. Furthermore it is clear by an elementary calculation that

$$\sum_{j=1}^{n} (\exp(i2q\pi/n))^j = 0$$

because of $0 < q/n < 1$. Hence by the definition of s_j again, we have

(4.14) $$\sum\nolimits_{odd} \cos(qs_j+c_*) = 0,$$

(4.15) $$\sum\nolimits_{even} \cos(qs_j+c_*) = 0,$$

where the first sum is taken over all odd integers j from 1 to $2n$, and the second sum is over all even integers j between 1 and $2n$.

5. **The maximum modulus principle.** In order to complete the proof of our theorem we need the following fact which is a slightly extension of a classic theorem on the removable singularity.

Lemma 7. *Let* $u(t)$ *and* $v(t)$ *be continuous functions of* t *for* $t \geqq t^*$ *satisfying the following conditions:*

a) *the modulus* $|u(t)|$ *and* $|v(t)|$ *are both strictly increasing and unbounded;*
b) *the argument of* $u(t)$ *increases steadily and becomes positively infinite while the argument of* $v(t)$ *decreases steadily and becomes negatively infinite.*

Let $G(z)$ *be a regular function in the annulus* $1 < |z| < +\infty$. *If the composite functions* $G(u(t))$ *and* $G(v(t))$ *are both bounded for* $t \geqq t^*$, *then* $G(z)$ *is regular at the point at infinity.*

Proof. By the condition a), we can consider the inverse functions of $|u(t)|$ and $|v(t)|$ on the interval $x \geqq x^*$, where x^* is a sufficiently large number. As before we denote these inverses by $p(x)$ and $q(x)$ respectively. Then it is clear that

$$|u(p(x))| = x, \quad |v(q(x))| = x \tag{5.1}$$

for $x \geqq x^*$. We next set

$$\begin{aligned} I(x) &= \arg u(p(x)), \\ J(x) &= \arg v(q(x)) \end{aligned} \tag{5.2}$$

for $x \geqq x^*$, where the branches of the argument are arbitrary. Then by the condition b), $I(x)$ increases steadily while $J(x)$ decreases steadily there. Hence the difference $I(x) - J(x)$ is strictly increasing indefinitely. Of course this difference is continuous. Therefore for all sufficiently large integers j, we can determine x_j uniquely so that

$$I(x_j) - J(x_j) = 2\pi j. \tag{5.3}$$

On the other hand it follows from (5.1) and (5.2) that

$$\begin{aligned} u(p(x)) &= x \exp(iI(x)), \\ v(q(x)) &= x \exp(iJ(x)) \end{aligned} \tag{5.4}$$

for $x \geqq x^*$. In particular by means of (5.3),

$$u(p(x_j)) = v(q(x_j)). \tag{5.5}$$

We now set the arcs

$$(5.6) \qquad \begin{aligned} L_j^+ &= \{u(t) : p(x_j) \leqq t \leqq p(x_{j+1})\}\,, \\ L_j^- &= \{v(t) : q(x_j) \leqq t \leqq q(x_{j+1})\}\,. \end{aligned}$$

With the aid of (5.4) the arcs L_j^+ and L_j^- are simple and lie in the annulus $x_j \leqq |z| \leqq x_{j+1}$. The arc L_j^+ winds around the origin in the positive direction while the arc L_j^- does in the negative direction. Furthermore by virtue of (5.5), the arcs L_j^+ and L_j^- have the same initial point and the same end point. Consequently because of

$$I(x_{j+1}) - I(x_j) + J(x_j) - J(x_{j+1}) = 2\pi,$$

the curves $L_j = L_j^+ - L_j^-$ are all simple closed and wind around the origin exactly once. Since each curve L_j lies entirely in the ring $x_j \leqq |z| \leqq x_{j+1}$ and the sequence $\{x_j\}$ converges to infinity with j, the sequence of the curves L_j converges to the point at infinity as j grows to infinity. On the other hand by assumption and the definition (5.6), the regular function $G(z)$ must be bounded uniformly on all the curves L_j. It therefore follows by the maximum modulus principle that the function $G(z)$ is bounded in a neighborhood of the point at infinity. Hence $G(z)$ is regular there. This completes the proof.

Considering the functions $1/u(t)$, $1/v(t)$ and $G(1/z)$, we can restate the above Lemma 7 as follows.

Lemma 8. *Let $u(t)$ and $v(t)$ be continuous functions of t for $t \geqq t^*$ satisfying the following conditions:*
a) *the modulus $|u(t)|$ and $|v(t)|$ both decrease steadily and converge to zero as t goes to infinity;*
b) *the argument of $u(t)$ increases steadily and becomes positively infinite while the argument of $v(t)$ decreases steadily and becomes negatively infinite.*

Let $G(z)$ be a regular function in the punctured unit disc $0 < |z| < 1$. Assume that the functions $G(u(t))$ and $G(v(t))$ are both bounded for $t \geqq t^*$. Then $G(z)$ is regular at the origin.

6. **Case 3.** Suppose that the case 3 occurs. Then the auxiliary function $f(z)$ has a pole at the point at infinity. Let q be its order of

the pole. Then $1 \leqq q < n$. We may assume that the function $f(z)$ has the form (4.4). We can thus make use of Lemma 4. By what was mentioned at the end of the section 4, especially (4.14) and (4.15), it is possible to find two both odd or both even integers h and k between 1 and $2n$ satisfying

$$\cos(qs_h + c_*) < 0, \quad \cos(qs_k + c_*) > 0 .$$

There are two cases to consider. We first assume that $(-1)^h a_*$ is positive. Then with the help of (4.9), the modulus $|w_h(t)|$ and $|w_k(t)|$ are both strictly increasing for $t \geqq t^*$, where t^* is a sufficiently large number. Of course they become positively infinite when t goes to infinity. More precisely we easily have

$$\lim_{t\to\infty} \frac{1}{t} \log |w_j(t)| = (-1)^j \frac{a_*}{b} > 0$$

for $j=h$ and $j=k$. Hence it follows from (1.7) that

$$\lim_{t\to\infty} \frac{(z_j(t))^n}{w_j(t)} = 0, \tag{6.1}$$

so that by means of (1.9),

$$\lim_{t\to\infty} \frac{A(w_j(t))}{w_j(t)} = 0 \tag{6.2}$$

for $j=h$ and $j=k$. This means that the function $A(z)/z$ is bounded on the curves $w_h(t)$ and $w_k(t)$. Furthermore it follows from (1.7) and (4.10) that

$$\lim_{t\to\infty} t^s \operatorname{Im} \frac{w_h'(t)}{w_h(t)} = I_h > 0, \tag{6.3}$$

$$\lim_{t\to\infty} t^s \operatorname{Im} \frac{w_k'(t)}{w_k(t)} = I_k < 0 \tag{6.4}$$

with $s = 1 - q/n$. Since $0 < s < 1$, the argument of $w_h(t)$ increases steadily and becomes positively infinite, while the argument of $w_k(t)$ decreases strictly without bound. Consequently by virtue of Lemma 4, the singularity of the function $A(z)/z$ at the point at infinity is removable. Further with the aid of (6.2) this function $A(z)/z$ takes the value zero at the point at infinity. Hereby the function $A(z)$ itself is regular there. However this is a contradiction by (1.9), because a is different from b.

We next suppose that $(-1)^h a_*$ is negative. This time with the help of (4.9) again, the modulus $|w_h(t)|$ and $|w_k(t)|$ both decrease steadily and converge to zero as t increases from a certain point t^* to infinity. Hence by making use of l'Hospital's rule, we deduce

$$\lim_{t\to\infty} \frac{1}{t} \log|w_j(t)| = (-1)^j \frac{a_*}{b} < 0$$

for $j=h$ and $j=k$. It thus follows from (1.7) that

$$\lim_{t\to\infty} (z_j(t))^n w_j(t) = 0, \tag{6.5}$$

so that by means of (1.9) again,

$$\lim_{t\to\infty} w_j(t)A(w_j(t)) = 0 \tag{6.6}$$

for $j=h$ and $j=k$. For the imaginary parts the estimates (6.3) and (6.4) remain valid. We now apply Lemma 8 with $u(t) = w_h(t)$, $v(t) = w_k(t)$ and $G(z) = zA(z)$. Consequently the function $zA(z)$ must be regular at the origin. As before by means of (6.6), this function $zA(z)$ takes the value zero at the origin, and hence the function $A(z)$ itself is regular there. However this contradicts (1.9) again. Accordingly we reject this case 3.

7. **Case 4.** In this section we treat the case 4 where the constant a is real and the auxiliary function $f(z)$ is regular at the point at infinity. Let us recall Lemma 5. Then by virtue of (1.7), the estimate (4.11) becomes

$$\lim_{t\to\infty} t \operatorname{Re} \frac{w_j'(t)}{w_j(t)} = \frac{b-a}{nb} \tag{7.1}$$

for $j=1, 2, \ldots, 2n$. It therefore follows that

$$\lim_{t\to\infty} \frac{\log|w_j(t)|}{\log t} = \frac{b-a}{nb} \tag{7.2}$$

for each j, provided that a is different from b. Let h and k be arbitrary integers satisfying $1 \leqq h \leqq 2n$, $1 \leqq k \leqq 2n$, $(-1)^h a < 0$ and $(-1)^k a > 0$. Then by virtue of (4.12), the functions $w_h(t)$ and $w_k(t)$ satisfy the condition b) of Lemmas 7 and 8.

Suppose first that $a < b$. Then from (7.1) and (7.2), each function $|w_j(t)|$ increases steadily and with some positive integer m,

$$\lim_{t\to\infty} \frac{t}{(w_j(t))^m} = 0. \tag{7.3}$$

Combining this (7.3) with (1.7) and (1.9) we at once have

$$\lim_{t\to\infty} \frac{A(w_j(t))}{(w_j(t))^m} = 0$$

for $j=1, 2, \ldots, 2n$. In particular the function $A(z)/z^m$ is bounded on the curves $w_j(t)$. Taking these facts into account we can apply Lemma 7 with $u(t) = w_h(t)$, $v(t) = w_k(t)$ and $G(z) = A(z)/z^m$. Then the function $A(z)$ is regular or has a pole at the point at infinity. Since $A(z)$ is not constant by assumption, with a suitable integer s and a nonzero constant C,

$$\lim_{z\to\infty} \frac{A(z)}{z^s} = C. \tag{7.4}$$

Consequently it follows from (1.7), (1.9) and (7.4) that

$$\lim_{t\to\infty} \frac{(w_j(t))^s}{t} = (-1)^j i \frac{b-a}{bC}$$

for $j=1, 2, \ldots, 2n$. Clearly the integer s must be positive, and hence each argument of $w_j(t)$ must converge to some finite value. This is absurd by (4.12).

Suppose next that $a > b$. Then by means of (7.1) and (7.2) again, all the modulus $|w_j(t)|$ decrease and converge to zero as t goes to infinity. Further with some positive integer m,

$$\lim_{t\to\infty} t(w_j(t))^m = 0 \tag{7.5}$$

for $j=1, 2, \ldots, 2n$. Hence it follows from (1.7), (1.9) and (7.5) that

$$\lim_{t\to\infty} (w_j(t))^m A(w_j(t)) = 0$$

for $j=1, 2, \ldots, 2n$. This time we apply Lemma 8 with $u(t) = w_h(t)$, $v(t) = w_k(t)$ and $G(z) = z^m A(z)$. Consequently the function $A(z)$ is regular or has a pole at the origin. Hence by the same reason as before, we can take some positive integer s such that

$$\lim_{t\to\infty} t(w_j(t))^s = C'$$

with a nonzero constant C'. Therefore every argument of $w_j(t)$ must converge to some finite value when t goes to infinity. This contradicts (4.12) again.

Accordingly the constant a must be equal to the constant b, and the auxiliary function $f(z)$ becomes $Q(z) - P(z)$. It therefore follows by assumption that the difference $P(z) - Q(z)$ is regular at the point at infinity. Consequently with the help of Lemma 4 [1], the functions $P(z)$ and $Q(z)$ differ by a constant. This is precisely what we wanted to prove.

8. **Case 5.** We finally consider the case 5. As before we may assume that the auxiliary function $f(z)$ has the form (4.4). For this case 5 we can make use of Lemma 6. Firstly let us note the relation

$$\cos 2x = -2\sin(x - \pi/4)\cos(x - \pi/4).$$

Then by what was mentioned at the end of the section 4, except for the case where $2q = n$ and $\cos 2c_* = 0$, we can find two integers h and k between 1 and $2n$ such that

$$(-1)^h a < 0, \quad \cos(qs_h + c_*) < 0, \tag{8.1}$$

$$(8.2) \qquad (-1)^k a > 0, \quad \cos(qs_k + c_*) < 0.$$

Let h be an integer between 1 and $2n$ satisfying (8.1). Then taking account of (1.7) and (8.1) we rewrite (4.13) and (4.12) as

$$(8.3) \qquad \lim_{t\to\infty} t^s \operatorname{Re} \frac{w_h'(t)}{w_h(t)} = R_h > 0,$$

$$(8.4) \qquad \lim_{t\to\infty} \operatorname{Im} \frac{w_h'(t)}{w_h(t)} = I_h > 0$$

respectively, where $s = 1 - q/n$. Note that $0 < s < 1$. Then the modulus $|w_h(t)|$ increases strictly and becomes infinite, and furthermore we obtain

$$\lim_{t\to\infty} t^{s-1} \log |w_h(t)| = \frac{R_h}{1-s}.$$

Hence by means of (1.7) and (1.9),

$$(8.5) \qquad \lim_{t\to\infty} \frac{A(w_h(t))}{w_h(t)} = 0.$$

Similarly if an integer k satisfies (8.2), then

$$(8.6) \qquad \lim_{t\to\infty} t^s \operatorname{Re} \frac{w_k'(t)}{w_k(t)} = R_k > 0,$$

$$(8.7) \qquad \lim_{t\to\infty} \operatorname{Im} \frac{w_k'(t)}{w_k(t)} = I_k < 0,$$

$$(8.8) \qquad \lim_{t\to\infty} \frac{A(w_k(t))}{w_k(t)} = 0.$$

Consequently with the aid of these (8.3)-(8.8), the functions $w_h(t)$, $w_k(t)$ and $A(z)/z$ satisfy the conditions of Lemma 7. It thus follows from Lemma 7 that the function $A(z)/z$ is regular at the point at infinity, and hence the function $A(z)$ is also regular there from (8.5) and (8.8). In particular $A(w_h(t))$ converges to some finite value as t tends to infinity. It therefore follows from (1.9) again that the constant a coincides with the constant b. Accordingly $f(z) = Q(z) - P(z)$ by the definition (1.12), so that the difference $P(z) - Q(z)$ has a pole of order q at the point at infinity. Here we recall the functional equation (1.2) and the definition (1.8). Then we have

$$P(z_h(t)) - Q(z_h(t)) = A(w_h(t)) - B(Q_*(z_h(t))).$$

Evidently by virtue of (1.4), the left hand side becomes infinite as t goes to infinity. On the other hand by the convergence of $A(w_h(t))$ and by the property (1.6), the right hand side must be bounded for $t \geqq 0$. This is a contradiction. Consequently the case 5 is absurd except for the subcase where $2q = n$ and $\cos 2c_* = 0$.

Hereafter we confine ourselves to this exceptional case. Let $F(z)$ be the function defined with (3.2). Since the constants a and b are real, we easily have

$$(8.9)\qquad \begin{aligned} &|1+zQ'(z)|^2 \operatorname{Im}(F(z)) \\ &= \operatorname{Im}(zP'(z) - zQ'(z)) + \operatorname{Im}(zP'(z)\overline{zf'(z)}). \end{aligned}$$

In order to estimate the imaginary part of $F(z)$ we need to estimate the second term of the right hand side. To do this we set the functions

$$(8.10)\qquad g(z) = ab(f(z))^2 - ac^2f(z) - bc^2P(z),$$

$$(8.11)\qquad R(z) = 2bf(z) - c^2.$$

Differentiating (8.10) we deduce

$$g'(z) + bc^2P'(z) = af'(z)R(z).$$

Hence it follows that

$$\begin{aligned} &a|R(z)|^2\, zP'(z)\overline{zf'(z)} \\ &= \overline{zg'(z)}zP'(z)R(z) - bc^2|zP'(z)|^2R(z) \end{aligned}$$

because c^2 is purely imaginary. Hereby

$$\begin{aligned}(8.12)\qquad & a|R(z)|^2\mathrm{Im}(zP'(z)\overline{zf'(z)}) \\ & = \mathrm{Im}(\overline{zg'(z)}zP'(z)R(z)) - 2b^2c'|zP'(z)|^2\mathrm{Re}(f(z)),\end{aligned}$$

where c' denotes the imaginary part of c^2, that is, $c^2 = ic'$. Furthermore it is clear from the definitions (1.12) and (8.10) that

$$(8.13)\qquad g(z) = ab(f(z))^2 - ac^2Q(z).$$

Since ab is real and c^2 is purely imaginary, it follows that

$$(8.14)\qquad \mathrm{Im}(g(z)) = 2ab\mathrm{Re}(f(z))\mathrm{Im}(f(z)) - ac'\mathrm{Re}(Q(z)).$$

Lemma 9. *If the function* $g(z)$ *is regular at the point at infinity, then*

$$(8.15)\qquad \lim_{t\to\infty}|z_j(t)|^n\,\mathrm{Re}\,\frac{w_j'(t)}{w_j(t)} = \frac{b-a}{nb^2}$$

for each j *satisfying* $\cos(qs_j+c_*) = 0$. *Furthermore if* $a = b$, *then*

$$(8.16)\qquad \lim_{t\to\infty}\frac{|z_j|^{n+q}}{\log|z_j|}\,\mathrm{Re}\,\frac{w_j'(t)}{w_j(t)} = (-1)^j\frac{ac^*}{4b^3}\sin(qs_j+c_*)$$

for each j *with* $\cos(qs_j+c_*) = 0$.

Proof. Since $g(z)$ is regular at the point at infinity, $z^2g(z)$ is bounded there. With the help of (4.4) and (8.11), the function $R(z)$ takes the form

$$(8.17)\qquad R(z) = (2bc+o(1))z^q$$

near the point at infinity. Hence by means of (1.3),

$$(8.18)\qquad \overline{zg'(z)}zP'(z)R(z) = O(z^{n+q-1})$$

there. Furthermore with the help of (1.4) and (1.6),

$$\lim_{t\to\infty} \frac{\mathrm{Re}(Q(z_j(t)))}{\log|z_j(t)|} = -1$$

for $j=1, 2, \ldots, 2n$. It thus follows from (8.14) that

$$\lim_{t\to\infty} \frac{\mathrm{Re}(f(z_j))\mathrm{Im}(f(z_j))}{\log|z_j(t)|} = -\frac{c'}{2b} \tag{8.19}$$

for $j=1, 2, \ldots, 2n$. On the other hand it is clear from (1.4), (1.5) and (4.4) that

$$\lim_{t\to\infty} \frac{\mathrm{Im}(f(z_j))}{|z_j(t)|^q} = c^*\sin(qs_j+c_*) \tag{8.20}$$

for $j=1, 2, \ldots, 2n$. Hence if an integer j between 1 and $2n$ satisfies

$$\cos(qs_j+c_*) = 0, \tag{8.21}$$

then the above (8.19) and (8.20) yield

$$\lim_{t\to\infty} \frac{|z_j|^q}{\log|z_j|} \mathrm{Re}(f(z_j)) = -\frac{c^*}{2b}\cos(qs_j-c_*),$$

because $c' = (c^*)^2 \sin 2c_*$ and $\cos(qs_j-c_*) = \sin 2c_*\sin(qs_j+c_*)$. Consequently we obtain

$$\lim_{t\to\infty} \frac{|z_jP'(z_j)|^2\mathrm{Re}(f(z_j))}{|z_j|^{2n-q}\log|z_j|} = -\frac{n^2a^2c^*}{2b}\cos(qs_j-c_*) \tag{8.22}$$

for each j satisfying (8.21). Here note that $2q = n$, and $2n-q = n+q$. Then by means of (8.12), (8.17), (8.18) and (8.22), we deduce

$$\lim_{t\to\infty} \frac{\mathrm{Im}(z_jP'(z_j)\overline{z_jf'(z_j)})}{|z_j|^q\log|z_j|} = \frac{n^2ac^*}{4b}\sin(qs_j+c_*), \tag{8.23}$$

and hence by virtue of (1.3) and (8.23),

$$\lim_{t\to\infty} |z_j(t)|^n \operatorname{Im}(F(z_j)) = (-1)^j \frac{b-a}{nb^2} \tag{8.24}$$

for each number j satisfying (8.21). Accordingly the estimate (8.15) follows at once from (4.2) and this (8.24).

Assume now that $a = b$. Then $f(z) = Q(z) - P(z)$ by the definition (1.12). Hence near the point at infinity

$$zQ'(z) - zP'(z) = zf'(z) = O(z^q).$$

It therefore follows from (8.9) and (8.23) that

$$\lim_{t\to\infty} \frac{|1+z_jQ'(z_j)|^2 \operatorname{Im}(F(z_j))}{|z_j|^q \log|z_j|} = \frac{n^2ac^*}{4b} \sin(sq_j+c_*)$$

for each j with (8.21). Combining this estimate with (1.3) and (4.2) we can obtain (8.16) immediately. This completes the proof.

Assume that the function $g(z)$ is regular at the point at infinity and that $a > b$. Let h be an integer between 1 and $2n$ which satisfies $\cos(qs_h + c_*) = 0$. Then by means of (1.7), (4.12) and (8.15), we obtain

$$\lim_{t\to\infty} t \operatorname{Re} \frac{w_h'(t)}{w_h(t)} = \frac{b-a}{nb} < 0, \tag{8.25}$$

$$\lim_{t\to\infty} \operatorname{Im} \frac{w_h'(t)}{w_h(t)} = (-1)^{h+1} \frac{a}{b}. \tag{8.26}$$

Hence the modulus $|w_h(t)|$ decreases steadily and converges to zero as t tends to infinity. Furthermore since

$$\lim_{t\to\infty} \frac{\log|w_h(t)|}{\log t} = \frac{b-a}{nb},$$

it follows from (1.7) and (1.9) that

$$\lim_{t\to\infty} (w_h(t))^m A(w_h(t)) = 0 \tag{8.27}$$

with some positive integer m. On the other hand it is possible to find an integer k between 1 and $2n$ such that $\cos(qs_k+c_*) > 0$. Here we remark that this integer k is odd or even according to whether the integer h is even or odd. With the help of (1.7), (4.12) and (4.13), we deduce

$$\lim_{t\to\infty} t^s \operatorname{Re} \frac{w_k'(t)}{w_k(t)} = R_k < 0, \tag{8.28}$$

$$\lim_{t\to\infty} \operatorname{Im} \frac{w_k'(t)}{w_k(t)} = (-1)^{k+1} \frac{a}{b}, \tag{8.29}$$

where $s = 1-q/n = 1/2$. Hence by making use of (1.7), (1.9) and (8.28), we also have

$$\lim_{t\to\infty} w_k(t) A(w_k(t)) = 0. \tag{8.30}$$

Consequently the functions $w_h(t)$, $w_k(t)$ and $z^m A(z)$ surely satisfy the conditions of Lemma 8. It therefore follows from Lemma 8 that the function $z^m A(z)$ is regular at the origin, so that the function $A(z)$ itself is regular there by virtue of (8.30). In particular $A(w_k(t))$ converges to the finite value $A(0)$ when t tends to infinity. This implies $a = b$ by (1.9). This is a contradiction.

Suppose next that the function $g(z)$ is regular at the point at infinity and that $a < b$. As before let h be an integer between 1 and $2n$ such that $\cos(qs_h+c_*) = 0$. Then with the help of (1.7), (4.12) and (8.15) again, we obtain

$$\lim_{t\to\infty} t \operatorname{Re} \frac{w_h'(t)}{w_h(t)} = \frac{b-a}{nb} > 0 \tag{8.31}$$

and (8.26). This time the modulus $|w_h(t)|$ increases steadily without bound and with some positive integer m,

$$\lim_{t\to\infty} \frac{A(w_h(t))}{(w_h(t))^m} = 0 \tag{8.32}$$

by means of (1.7) and (1.9) again. Let k be an integer between 1 and $2n$ such that $\cos(qs_k+c_*) < 0$. By what was mentioned at the end of the section 4, such an integer k surely exists and it must be odd or even according to whether the integer h is even or odd. With the aid of (1.7), (4.12) and (4.13) we also have

$$\lim_{t\to\infty} t^s \operatorname{Re} \frac{w_k'(t)}{w_k(t)} = R_k > 0 \tag{8.33}$$

and (8.29), where $s = 1/2$. It therefore follows from (1.7), (1.9) and (8.33) that

$$\lim_{t\to\infty} \frac{A(w_k(t))}{w_k(t)} = 0. \tag{8.34}$$

This time we can apply Lemma 7. Then the function $A(z)$ is regular at the point at infinity by means of (8.34). Consequently $a = b$ by virtue of (1.9) again. This is a contradiction.

Finally suppose that $g(z)$ is regular at the point at infinity and that $a = b$. Let j be an integer between 1 and $2n$ which satisfies $\cos(qs_j + c_*) = 0$. Then by means of (1.7), the estimate (8.16) becomes

$$\lim_{t\to\infty} \frac{t^{3/2}}{\log t} \operatorname{Re} \frac{w_j'(t)}{w_j(t)} = R_j,$$

where R_j is a nonzero constant. It therefore follows that $\log|w_j(t)|$ converges to some finite value as t goes to infinity. This means that the function $A(w_j(t))$ must be bounded for $t \geqq 0$. Consequently by the functional equation (1.2) and by the property (1.6), the function $P(z_j(t)) - Q(z_j(t))$ is also bounded for $t \geqq 0$. On the other hand by the assumption $a = b$, the auxiliary function $f(z)$ coincides with $Q(z) - P(z)$. Hence the difference $P(z) - Q(z)$ has a pole of order q at the point at infinity. In particular by means of (1.4), $P(z_j(t)) - Q(z_j(t))$ becomes infinite as t goes to infinity. This is a contradiction. Accordingly the auxiliary function $g(z)$ defined by (8.10) is not regular at the point at infinity.

We next consider the case where the function $g(z)$ has a pole at the point at infinity. Let r stand for its order of the pole. Then $1 \leqq r < n$, and the function $g(z)$ takes the form

$$(8.35) \qquad g(z) = (d+o(1))z^r,$$

where d is a nonzero constant. With the help of (1.3), (8.17) and this (8.35), we can easily have

$$(8.36) \qquad \begin{aligned} &\lim_{t\to\infty} \frac{\operatorname{Im}(\overline{z_j g'(z_j)} z_j P'(z_j) R(z_j))}{|z_j(t)|^{n+q+r}} \\ &= (-1)^{j+1} 2nrabc^* d^* \cos((q-r)s_j + c_* - d_*) \end{aligned}$$

for $j=1, 2, \ldots, 2n$, where $d^* = |d|$ and $d_* = \arg d$. On the other hand by means of (8.14) and (8.35) again,

$$\lim_{t\to\infty} \frac{\operatorname{Re}(f(z_j))\operatorname{Im}(f(z_j))}{|z_j(t)|^r} = \frac{d^*}{2ab} \sin(rs_j + d_*),$$

so that (8.20) implies

$$(8.37) \qquad \lim_{t\to\infty} \frac{\operatorname{Re}(f(z_j))}{|z_j|^{r-q}} = \frac{d^*}{2abc^*} \cos((q-r)s_j + c_* - d_*)$$

for each number j satisfying (8.21). Consequently we obtain

$$(8.38) \qquad \begin{aligned} &\lim_{t\to\infty} \frac{|z_j P'(z_j)|^2 \operatorname{Re}(f(z_j))}{|z_j(t)|^{2n-q+r}} \\ &= \frac{n^2 a d^*}{2bc^*} \cos((q-r)s_j + c_* - d_*) \end{aligned}$$

for each j which satisfies (8.21). Here we remark that $2n-q+r = n+q+r$ and $\sin 2c_* = (-1)^{j+1}$ provided j satisfies (8.21). Indeed $\cos(qs_j + c_*) = 0$ implies

$$-1 = \cos(2qs_j + 2c_*) = \cos(ns_j + 2c_*) = (-1)^j \sin 2c_*,$$

because $2q = n$, $\cos 2c_* = 0$ and $\sin ns_j = (-1)^{j+1}$. Taking (8.12), (8.17), (8.36) and (8.38) into account we thus obtain

$$\lim_{t\to\infty}\frac{\operatorname{Im}(z_jP'(z_j)\overline{z_jf'(z_j)})}{|z_j(t)|^{q+r}} \tag{8.39}$$
$$= (-1)^j\frac{(n-2r)nd^*}{4bc^*}\cos((q-r)s_j+c_*-d_*)$$

for each number j satisfying (8.21). We are now in a position to prove the next final lemma.

Lemma 10. *Suppose that the function $g(z)$ has a pole at the point at infinity and takes the form* (8.35). *If the order r is greater than q, then*

$$\lim_{t\to\infty}|z_j(t)|^{n+q-r}\operatorname{Re}\frac{w_j'(t)}{w_j(t)} \tag{8.40}$$
$$=\frac{(n-2r)d^*}{4nb^3c^*}\cos((q-r)s_j+c_*-d_*)$$

for each number j satisfying $\cos(qs_j+c_*)=0$. *If $r\leqq q$, then the estimate* (8.15) *remains valid for each j which satisfies* $\cos(qs_j+c_*)=0$. *Furthermore if $r\leqq q$, then for every number j satisfying* $\cos(qs_j+c_*)=0$, *the real part of $f(z_j(t))$ converges to some finite value when t goes to infinity.*

Proof. Assume first that $r>q$. Then $q+r>2q=n$. It thus follows from (1.3), (8.9) and (8.39) that

$$\lim_{t\to\infty}|z_j(t)|^{n+q-r}\operatorname{Im}(F(z_j)) \tag{8.41}$$
$$=(-1)^j\frac{(n-2r)d^*}{4nb^3c^*}\cos((q-r)s_j+c_*-d_*)$$

for every j that satisfies (8.21). On combining this (8.41) with (4.2) we hence have the desired (8.40).

Next we suppose that $r\leqq q$. Then $q+r\leqq n$, and hence by means of (8.39) again,

$$\lim_{t\to\infty} \frac{\operatorname{Im}(z_j P'(z_j)\overline{z_j f'(z_j)})}{|z_j(t)|^n} = 0$$

for each j satisfying (8.21). Consequently the relation (8.9) yields

$$\lim_{t\to\infty} \frac{|1+z_j Q'(z_j)|^2}{|z_j(t)|^n} \operatorname{Im}(F(z_j)) = n(a-b)\sin ns_j,$$

so that by means of (1.3),

$$\lim_{t\to\infty} |z_j(t)|^n \operatorname{Im}(F(z_j)) = (-1)^{j+1} \frac{a-b}{nb^2}$$

for each number j satisfying (8.21). The desired (8.15) follows directly from this estimate and (4.2).

The last statement is an immediate consequence of the above (8.37) because of $r-q \leqq 0$. This completes the proof of Lemma 10.

In order to treat the exceptional case completely we must consider the case where the auxiliary function $g(z)$ has a pole at the point at infinity. Let r stand for the order of the pole.

Suppose first that $r > q$. Then $0 < r-q < n-q = q$. Hence there exists an integer h between 1 and $2n$ which satisfies

$$\cos(qs_h + c_*) = 0, \quad \cos((q-r)s_h + c_* - d_*) < 0$$

simultaneously. It thus follows from (1.7) and (8.40) that

$$\lim_{t\to\infty} t^s \operatorname{Re} \frac{w_h'(t)}{w_h(t)} = R_h > 0$$

with $s = (n+q-r)/n$. Note that $1/2 < s < 1$ by assumption. Hereby with the aid of (1.7) and (1.9),

$$\lim_{t\to\infty} \frac{A(w_h(t))}{w_h(t)} = 0.$$

Let k denote an integer between 1 and $2n$ such that $\cos(qs_k + c_*) < 0$. Then the function $w_k(t)$ satisfies (8.33) and (8.34). Hereby the functions

$w_h(t)$, $w_k(t)$ and $A(z)/z$ satisfy the conditions of Lemma 7. Consequently by exactly the same reason as before we arrive at a contradiction.

Suppose next that $r \leqq q$. Then by Lemma 10, the previous estimate (8.15) still holds. Hence if a is different from b, by exactly the same manner as in the case where $g(z)$ is regular at the point at infinity, we also arrive at a contradiction. Therefore it only remains to consider the case where $r \leqq q$ and $a = b$. If $a = b$, then $f(z) = Q(z) - P(z)$ by the definition (1.12). Hence it is clear from (1.1) that

$$Q_*(z) = P_*(z)\exp(f(z)).$$

In particular by the property (1.6) and by the definition (1.8),

$$\log|w_j(t)| + \mathrm{Re}(f(z_j(t))) = \log R$$

for real values of t with $t \geqq 0$. Here we recall Lemma 10. Then for each number j satisfying $\cos(qs_j + c_*) = 0$, $\log|w_j(t)|$ converges to some finite value as t tends to infinity. Consequently $A(w_j(t))$ must be bounded for $t \geqq 0$. It therefore follows from the functional equation (1.2) and the property (1.6) that $f(z_j(t))$ is also bounded for $t \geqq 0$. This is impossible since the auxiliary function $f(z)$ has a pole of order q at the point at infinity.

Accordingly the exceptional case never happens. Hence the case 5 is impossible. The proof of our theorem is now complete.

References

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