

## On Certain Functional Equations, I

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Let  $A(z)$ ,  $B(z)$ ,  $P(z)$  and  $Q(z)$  be nonconstant regular functions in the annulus  $0 < |z| < +\infty$ . Suppose that the functional equation

$$P(z) - Q(z) = A(ze^{P(z)}) - B(ze^{Q(z)})$$

holds in this annulus. Then what can be said about these four functions? In our previous paper [2] we considered this functional equation under some additional conditions. For instance the following results were proved.

**Proposition A.** *Suppose that the functions  $P(z)$  and  $Q(z)$  have poles at the point at infinity, and that the difference  $P(z) - Q(z)$  is regular there. Then  $P(z)$  and  $Q(z)$  differ by a constant.*

**Proposition B.** *Suppose that  $Q(z)$  has a pole of order  $n$  at the point at infinity, and that  $P(z)/z^n$  is regular there. Suppose further that the function  $A(z)$  has a nonessential singularity at the point at infinity. Then  $P(z)$  and  $Q(z)$  differ by a constant.*

The purpose of this article is to show the next theorem.

**Theorem.** *Assume that the functions  $P(z)$  and  $Q(z)$  have poles at the point at infinity. Then the order of the pole of  $P$  is equal to that of  $Q$ .*

1. **Preliminary.** Let  $P(z)$  and  $Q(z)$  be nonconstant regular functions in the annulus  $0 < |z| < +\infty$  which have poles at the point at infinity. Then with two natural numbers  $m$  and  $n$ ,

$$(1.1) \quad P(z) = (a + o(1))z^m, \quad Q(z) = (b + o(1))z^n$$

near the point at infinity, where  $a$  and  $b$  are nonzero constants. Hereafter for the sake of simplicity we assume that  $1 \leq m < n$  and the constant  $b$  is real positive. The starting point of our consideration is the following lemma which was proved in [2].

**Lemma 1.** *There exist  $2n$  differentiable functions  $z_j(t)$  ( $j = 1, \dots, 2n$ ) of  $t$  for  $t \geq 0$  such that*

$$(1.2) \quad \lim_{t \rightarrow \infty} |z_j(t)| = +\infty,$$

$$(1.3) \quad \lim_{t \rightarrow \infty} \arg z_j(t) = (2j-1)\pi/2n,$$

$$(1.4) \quad z_j(t) \exp(Q(z_j(t))) = R \exp((-1)^{j+1} it)$$

for  $t \geq 0$ , where  $R$  is a suitable real positive constant.

It follows from (1.4) that

$$(1.5) \quad \arg z_j(t) + \operatorname{Im} Q(z_j(t)) = (-1)^{j+1} t + 2m_j \pi,$$

$$(1.6) \quad \log |z_j(t)| + \operatorname{Re} Q(z_j(t)) = \log R$$

for  $t \geq 0$  and  $1 \leq j \leq 2n$ , where  $m_j$  are suitable integers. Hence by using (1.1)-(1.3) and (1.5), we obtain

$$(1.7) \quad \lim_{t \rightarrow \infty} t^{-1} |z_j(t)|^n = b^{-1}$$

for each  $j$  with  $1 \leq j \leq 2n$ . Furthermore by differentiating (1.4), we have

$$(1.8) \quad z_j'(t)(1 + z_j(t)Q'(z_j(t))) = (-1)^{j+1} i z_j(t)$$

for  $t \geq 0$  and  $1 \leq j \leq 2n$ .

With these functions  $z_j(t)$  let us set

$$(1.9) \quad w_j(t) = z_j(t) \exp(P(z_j(t)))$$

for  $t \geq 0$  and  $j = 1, \dots, 2n$ . Our task is to investigate the nature of these functions  $w_j(t)$  precisely. Of course they are differentiable and satisfy

$$\frac{w_j'(t)}{w_j(t)} = (1 + z_j(t)P'(z_j(t))) \frac{z_j'(t)}{z_j(t)},$$

so that by means of (1.8)

$$(1.10) \quad \frac{w_j'(t)}{w_j(t)} = (-1)^{j+1} i \frac{1 + z_j(t)P'(z_j(t))}{1 + z_j(t)Q'(z_j(t))}$$

for  $t \geq 0$ ,  $j=1, \dots, 2n$ . On the other hand by the asymptotic behaviors (1.1) of  $P(z)$  and  $Q(z)$ ,

$$\frac{1 + zP'(z)}{1 + zQ'(z)} = \left( \frac{ma}{nb} + o(1) \right) z^{m-n}$$

near the point at infinity. It thus follows from (1.2), (1.3) and (1.10) that

$$\lim_{t \rightarrow \infty} |z_j(t)|^{n-m} \frac{w_j'(t)}{w_j(t)} = \frac{ma}{nb} \exp(ims_j)$$

for  $1 \leq j \leq 2n$ , where  $s_j = (2j-1)\pi/2n$ . Consequently we obtain

$$(1.11) \quad \lim_{t \rightarrow \infty} |z_j(t)|^{n-m} \operatorname{Re} \frac{w_j'(t)}{w_j(t)} = \frac{ma_*}{nb} \cos(ms_j + a^*),$$

$$(1.12) \quad \lim_{t \rightarrow \infty} |z_j(t)|^{n-m} \operatorname{Im} \frac{w_j'(t)}{w_j(t)} = \frac{ma_*}{nb} \sin(ms_j + a^*)$$

for  $1 \leq j \leq 2n$ , where  $a_* = |a|$  and  $a^* = \arg a$ .

The next main lemma on the logarithmic derivatives  $w_j'(t)/w_j(t)$  plays an important role in the proof of our theorem.

**Main Lemma.** *Among the  $2n$  functions  $w_j(t)$ , it is possible to choose two functions  $w_h(t)$  and  $w_k(t)$  such that*

$$(1.13) \quad \lim_{t \rightarrow \infty} t^\alpha \operatorname{Re} \frac{w_h'(t)}{w_h(t)} = R_h,$$

$$(1.14) \quad \lim_{t \rightarrow \infty} t^\beta \operatorname{Re} \frac{w'_k(t)}{w_k(t)} = R_k,$$

$$(1.15) \quad \lim_{t \rightarrow \infty} t^\gamma \operatorname{Im} \frac{w'_h(t)}{w_h(t)} = I_h,$$

$$(1.16) \quad \lim_{t \rightarrow \infty} t^\gamma \operatorname{Im} \frac{w'_k(t)}{w_k(t)} = I_k,$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are real numbers with  $0 < \alpha \leq 1$ ,  $0 < \beta \leq 1$  and  $0 < \gamma < 1$ , and  $R_h$ ,  $R_k$ ,  $I_h$  and  $I_k$  are nonzero real constants with  $R_h R_k > 0$ ,  $I_h > 0$  and  $I_k < 0$ .

Furthermore if  $\alpha = 1$ , then  $R_h = 1/n$ , and  $R_k = 1/n$  when  $\beta = 1$ .

The proof of this main lemma needs a little bit complicated process, so we divide the proof into three parts.

First of all we shall show the main lemma in the case when  $2m \neq n$  and  $3m \neq 2n$ . For this case we require the next elementary fact whose proof we omit.

**Lemma 2.** *Let  $m$  and  $n$  be natural numbers, and let  $l$  be the greatest common measure of  $m$  and  $2n$ . Then the set of complex numbers*

$$\{\exp(imj\pi/n) : j=1, 2, \dots, 2n\}$$

*agrees with the set*

$$\{\exp(ilj\pi/n) : j=1, 2, \dots, 2n\}.$$

Now let us consider the case in which  $2m \neq n$  and  $3m \neq 2n$ . In this case the greatest common measure  $l$  of  $m$  and  $2n$  satisfies  $2l < n$ . In fact if  $2l \geq n$ , then  $l \leq m < n \leq 2l$  by the assumption  $m < n$ . Hence  $1 \leq m/l < 2$ , so that  $l = m$ . Therefore  $2 < 2n/m \leq 4$ . Hereby  $2n/m = 3$  or  $2n/m = 4$  because  $2n/m = 2n/l$  is an integer. Consequently if  $3m \neq 2n$  and  $2m \neq n$ , then  $2l < n$ . By this fact and by virtue of Lemma 2, we can easily find two numbers  $h$  and  $k$  among the integers from 1 to  $2n$  such that

$$(1.17) \quad \begin{aligned} \cos(ms_h + a^*) > 0, \quad \cos(ms_k + a^*) > 0, \\ \sin(ms_h + a^*) > 0, \quad \sin(ms_k + a^*) < 0. \end{aligned}$$

Hence using (1.11) and (1.12), together with (1.7), we at once obtain (1.13)-(1.16) with  $\alpha = \beta = \gamma = 1 - m/n$ ,  $R_h > 0$  and  $R_k > 0$ . Accordingly we have proved the main lemma in this case.

**2. The proof of Main Lemma in the case  $2m=n$ .** In this section we shall show the main lemma in the case  $2m=n$ . Since  $\sin ns_j = (-1)^{j+1}$  and  $2m=n$ ,

$$(2.1) \quad (-1)^{j+1} \cos 2a^* = 2 \sin(ms_j + a^*) \cos(ms_j + a^*)$$

for  $1 \leq j \leq 2n$ . Hence if  $\cos 2a^* \neq 0$ ,  $\sin(ms_j + a^*)$  and  $\cos(ms_j + a^*)$  are all different from zero. Since  $4ms_j = (2j-1)\pi$ , it is clear that  $\cos(ms_{j+2} + a^*) = -\cos(ms_j + a^*)$  and

$$\cos(ms_{j+1} + a^*) \cos(ms_j + a^*) = -\sin(ms_{j+1} + a^*) \sin(ms_j + a^*)$$

for  $1 \leq j \leq 2n$ . Hereby if  $\cos 2a^* \neq 0$ , we can find two numbers  $h$  and  $k$  among the integers from 1 to  $2n$  which satisfy (1.17). Consequently unless  $\cos 2a^* = 0$ , the main lemma follows immediately in the case  $2m=n$ .

2.1. Hereafter we assume that  $2m=n$  and  $\cos 2a^* = 0$ . By (2.1), for each  $j$  with  $1 \leq j \leq 2n$ , either  $\sin(ms_j + a^*) = 0$  or  $\cos(ms_j + a^*) = 0$ . Furthermore since

$$(-1)^j \sin 2a^* = \cos^2(ms_j + a^*) - \sin^2(ms_j + a^*)$$

for  $1 \leq j \leq 2n$ , if  $\sin 2a^* = 1$ , then  $\cos(ms_j + a^*) = 0$  or  $\sin(ms_j + a^*) = 0$  according to whether  $j$  is odd or even. On the contrary if  $\sin 2a^* = -1$ ,  $\cos(ms_j + a^*) = 0$  or  $\sin(ms_j + a^*) = 0$  according to whether  $j$  is even or odd.

Let us set the auxiliary function

$$(2.2) \quad f(z) = a^2 Q(z) - b(P(z))^2.$$

Then this function  $f(z)$  is regular or has a pole at the point at infinity, so that with a suitable integer  $q$ ,

$$(2.3) \quad f(z) = cz^q + O(z^{q-1})$$

near the point at infinity, where  $c$  is a nonzero constant. By (1.1) and the assumption  $2m=n$ , the integer  $q$  must be less than  $n$ . Evidently since  $\cos 2a^* = 0$  by assumption,

$$\operatorname{Im} f(z) = a_*^2 \sin 2a^* \operatorname{Re}(Q(z)) - 2b \operatorname{Re}(P(z)) \operatorname{Im}(P(z)).$$

Hence it follows from (1.2), (1.3), (1.6) and (2.3) that

$$(2.4) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re}(P(z_j)) \operatorname{Im}(P(z_j))}{|z_j(t)|^q} = -\frac{c_*}{2b} \sin(qs_j + c^*)$$

when  $q \geq 1$ , and if  $q \leq 0$ ,

$$(2.5) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re}(P(z_j)) \operatorname{Im}(P(z_j))}{\log |z_j(t)|} = -\frac{a_*^2}{2b} \sin 2a^*$$

for each  $j$ , where  $c_* = |c|$  and  $c^* = \arg c$ . Therefore with the help of (1.1)-(1.3), if  $q \geq 1$ , the above (2.4) becomes

$$(2.6) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re} P(z_j(t))}{|z_j(t)|^{q-m}} = -\frac{c_*}{2a_*b} \cos(qs_j - ms_j - a^* + c^*)$$

for each  $j$  such as  $\cos(ms_j + a^*) = 0$ , and

$$(2.7) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Im} P(z_j(t))}{|z_j(t)|^{q-m}} = -\frac{c_*}{2a_*b} \sin(qs_j - ms_j - a^* + c^*)$$

for each  $j$  with  $\sin(ms_j + a^*) = 0$ . For the case  $q \leq 0$ ,

$$(2.8) \quad \lim_{t \rightarrow \infty} |z_j(t)|^m \frac{\operatorname{Re} P(z_j(t))}{\log |z_j(t)|} = -\frac{a_*}{2b} \cos(ms_j - a^*)$$

for each  $j$  satisfying  $\cos(ms_j + a^*) = 0$ .

On the other hand it is clear that

$$f'(z) = a^2 Q'(z) - 2bP(z)P'(z)$$

from the definition (2.2). It thus follows that

$$(2.9) \quad a^2 \overline{zP'(z)} zQ'(z) = \overline{zP'(z)} z f'(z) + 2b|zP'(z)|^2 P(z).$$

Hereby if  $q \geq 1$ , using (2.3), (2.6), (2.7) and (2.9), together with (1.1)-(1.3), we obtain

$$(2.10) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re}(z_j P'(z_j) \overline{z_j Q'(z_j)})}{|z_j(t)|^{m+q}} = \frac{c_*}{a_*} m(m-q) \cos(qs_j - ms_j + a^* + c^*)$$

for each  $j$  such that  $\sin(ms_j + a^*) = 0$ , and

$$(2.11) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Im}(z_j P'(z_j) \overline{z_j Q'(z_j)})}{|z_j(t)|^{m+q}} = \frac{c_*}{a_*} m(q-m) \sin(qs_j - ms_j + a^* + c^*)$$

for every  $j$  satisfying  $\cos(ms_j + a^*) = 0$ .

2.2. We now prove the following.

**Lemma 3.** *Let  $q$  be the integer defined by (2.3). If  $q > m$ , then*

$$(2.12) \quad \begin{aligned} & \lim_{t \rightarrow \infty} |z_j(t)|^{m+n-q} \operatorname{Im} \frac{w_j'(t)}{w_j(t)} \\ &= \frac{(m-q)c_*}{2na_*b^2} \sin(qs_j - ms_j - a^* + c^*) \end{aligned}$$

for each  $j$  satisfying  $\sin(ms_j + a^*) = 0$ . If  $q \leq m$ , then

$$(2.13) \quad \lim_{t \rightarrow \infty} |z_j(t)|^n \operatorname{Re} \frac{w_j'(t)}{w_j(t)} = \frac{1}{nb}$$

for each  $j$  with  $\cos(ms_j + a^*) = 0$ .

*Proof.* Suppose first that  $q > m$ . Then by means of (1.10) and (2.10),

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{|1 + z_j Q'(z_j)|^2}{|z_j(t)|^{m+q}} \operatorname{Im} \frac{w_j'(t)}{w_j(t)} \\ = (-1)^{j+1} \frac{c^*}{a_*} m(m-q) \cos(ms_j - qs_j - a^* - c^*), \end{aligned}$$

so that with the help of (1.1),

$$\begin{aligned} (2.14) \quad \lim_{t \rightarrow \infty} |z_j(t)|^{m+n-q} \operatorname{Im} \frac{w_j'(t)}{w_j(t)} \\ = (-1)^j \frac{(q-m)c^*}{2na_*b^2} \cos(qs_j - ms_j + a^* + c^*), \end{aligned}$$

for each  $j$  such that  $\sin(ms_j + a^*) = 0$ . Here note that  $\sin 2a^* = (-1)^j$  for every number  $j$  with  $\sin(ms_j + a^*) = 0$ . Hereby the above (2.14) becomes the desired (2.12).

Suppose next that  $q \leq m$ . If  $1 \leq q \leq m$ , by means of (2.11),

$$(2.15) \quad \lim_{t \rightarrow \infty} |z_j(t)|^{-n} \operatorname{Im}(z_j P'(z_j) \overline{z_j Q'(z_j)}) = 0$$

for each  $j$  such that  $\cos(ms_j + a^*) = 0$ . For the case  $q \leq 0$  we can also deduce (2.15) with the help of (2.3), (2.8) and (2.9). It therefore follows from (1.1)-(1.3) and (1.10) that

$$\lim_{t \rightarrow \infty} \frac{|1 + z_j Q'(z_j)|^2}{|z_j(t)|^n} \operatorname{Re} \frac{w_j'(t)}{w_j(t)} = nb$$

for every  $j$  satisfying  $\cos(ms_j + a^*) = 0$ . Hereby the desired (2.13) follows at once. This completes the proof.

2.3. Suppose that  $q > m$ . Since  $2m = n$  by assumption,

$$(2.16) \quad ms_j + a^* = \frac{2j-1}{4} \pi + a^*$$



for  $1 \leq j \leq 2n$ . Hence there uniquely exists an integer  $r$  between 1 and 4 such that

$$(2.17) \quad \cos(ms_r + a^*) = 1.$$

With this integer  $r$ , let us define the set of integers

$$J_r = \{4l + r : 0 \leq l \leq m-1\}.$$

Then it is clear from (2.16) and (2.17) that

$$(2.18) \quad \cos(ms_j + a^*) = 1$$

for every  $j$  of the set  $J_r$ . Since  $m < q < n$ ,  $0 < 2q - 2m < n$ , so that

$$\sum_{l=0}^{m-1} \exp(i4(q-m)l\pi/n) = 0.$$

By this fact we at once have

$$\sum \sin(qs_j - ms_j - a^* + c^*) = 0,$$

where the sum runs over the integers  $j$  of the set  $J_r$ . Hence it is possible to find two integers  $h$  and  $k$  of the set  $J_r$  such that

$$(2.19) \quad \begin{aligned} \sin(qs_h - ms_h - a^* + c^*) &< 0, \\ \sin(qs_k - ms_k - a^* + c^*) &> 0 \end{aligned}$$

except for the case when  $\sin(qs_j - ms_j - a^* + c^*) = 0$  for all  $j$  of  $J_r$ . Consequently except for this case, by virtue of (1.7), (1.11), (2.12), (2.18) and (2.19), we obtain the desired (1.13)-(1.16) with  $\alpha = \beta = 1/2$ ,  $R_h = R_k > 0$  and  $1/2 < \gamma < 1$ .

Now let us consider the above exceptional case in which

$$\sin(qs_j - ms_j - a^* + c^*) = 0$$

for all  $j$  of the set  $J_r$ . In this case the integer  $q$  must be

$$4(q-m)=n \text{ and}$$

$$(2.20) \quad \sin(qs_r - ms_r - a^* + c^*) = 0.$$

Let  $p$  be the integer between 1 and 4 satisfying  $|p-r|=2$ . Then it is clear that  $|ms_p - ms_r| = \pi$  by (2.16), so that

$$\cos(ms_p + a^*) = -1$$

from (2.17). Furthermore  $(q-m)|s_p - s_r| = \pi/2$ , so that

$$|\sin(qs_p - ms_p - a^* + c^*)| = 1$$

by (2.20). Accordingly it is possible to take two integers  $h$  and  $k$  which satisfy (2.19) and

$$(2.21) \quad \cos(ms_h + a^*) = \cos(ms_k + a^*) = -1.$$

Hence taking (1.7), (1.11), (2.12), (2.19) and (2.21) into account, we also obtain the desired (1.13)-(1.16) with  $\alpha = \beta = 1/2$ ,  $1/2 < \gamma < 1$  and  $R_h = R_k < 0$ .

2.4. It remains to consider the case  $q \leq m$ . For this case let us recall (1.12) and (2.13) of Lemma 3. Surely we can take two integers  $h$  and  $k$  such that

$$\sin(ms_h + a^*) = 1, \quad \sin(ms_k + a^*) = -1.$$

Hence with the help of (1.7) and (1.12),

$$\lim_{t \rightarrow \infty} t^{1/2} \operatorname{Im}(w_h'(t)/w_h(t)) = a_* b^{-1/2}/2,$$

$$\lim_{t \rightarrow \infty} t^{1/2} \operatorname{Im}(w_k'(t)/w_k(t)) = -a_* b^{-1/2}/2.$$

Furthermore with the help of (1.7) and (2.13), we obtain

$$\lim_{t \rightarrow \infty} t \operatorname{Re}(w_h'(t)/w_h(t)) = \lim_{t \rightarrow \infty} t \operatorname{Re}(w_k'(t)/w_k(t)) = 1/n.$$

These are just the desired (1.13)-(1.16). We have now proved

the main lemma in the case  $2m=n$  completely.

**3. The proof of Main Lemma in the case  $3m=2n$ .** Suppose that  $3m=2n$ . Then  $3ms_j=2ns_j=(2j-1)\pi$ , so that

$$(3.1) \quad \cos 3a^* = (4\sin^2(ms_j + a^*) - 1)\cos(ms_j + a^*)$$

for  $1 \leq j \leq 2n$ . Hence if  $\cos 3a^* \neq 0$ ,  $\cos(ms_j + a^*)$  never vanishes for any  $j$  with  $1 \leq j \leq 2n$ . Hereby we can find two integers  $h$  and  $k$  among 1, 2 and 3 such that  $\cos(ms_h + a^*)$  and  $\cos(ms_k + a^*)$  are both positive or negative simultaneously. Evidently  $|ms_h - ms_k| = 2\pi/3$  or  $4\pi/3$ . Therefore

$$\cos(ms_h + a^*)\cos(ms_k + a^*) + \sin(ms_h + a^*)\sin(ms_k + a^*) = -1/2,$$

so that  $\sin(ms_h + a^*)\sin(ms_k + a^*) < 0$ . By this fact, the desired results (1.13)-(1.16) follow at once from (1.7), (1.11) and (1.12) unless  $\cos 3a^* = 0$ . Note that the real quantities  $\alpha$ ,  $\beta$  and  $\gamma$  are all equal to  $1 - m/n = 1/3$  in this case.

3.1. From now on we confine ourselves to the case  $3m=2n$  and  $\cos 3a^* = 0$ . By (3.1), either  $\cos(ms_j + a^*) = 0$  or  $4\sin^2(ms_j + a^*) = 1$  for each  $j$  with  $1 \leq j \leq 2n$ . Since

$$(3.2) \quad ms_j + a^* = \frac{2j-1}{3}\pi + a^*,$$

there surely exists an integer  $r$  among 1, 2 and 3 such that  $\cos(ms_r + a^*) = 0$ . Hence by (3.2),  $\cos(ms_j + a^*) = 0$  for each  $j$  such that the difference  $j - r$  is a multiple of 3.

Let us consider the function  $g(z)$  defined with

$$(3.3) \quad g(z) = a^3(Q(z))^2 - b^2(P(z))^3.$$

Then this function  $g(z)$  is regular or has a pole at the point at infinity. Hence with a suitable integer  $p$ ,

$$(3.4) \quad g(z) = dz^p + O(z^{p-1})$$

near the point at infinity, where  $d$  is a nonzero constant. Since  $3m=2n$  by assumption, the integer  $p$  must be less than  $2n$ . It thus follows from (1.2), (1.3) and (3.4) that

$$(3.5) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re} g(z_j(t))}{|z_j(t)|^p} = d_* \cos(ps_j + d^*)$$

for each  $j$  with  $1 \leq j \leq 2n$ , where  $d_* = |d|$  and  $d^* = \arg d$ . On the other hand by the definition (3.3),

$$(3.6) \quad \operatorname{Re} g(z) + a_*^3 \sin 3a^* \operatorname{Im}(Q(z))^2 + b^2 \operatorname{Re}(P(z))^3 = 0.$$

Here let us recall the relation (1.6). Since

$$\operatorname{Im}(Q(z))^2 = 2\operatorname{Re}(Q(z))\operatorname{Im}(Q(z)),$$

it then follows from (1.1)-(1.3) that

$$(3.7) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Im}(Q(z_j(t)))^2}{|z_j|^n \log|z_j|} = (-1)^j 2b$$

for  $1 \leq j \leq 2n$ . Taking these (3.5)-(3.7) into account, we therefore have for each  $j$ ,

$$(3.8) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re}(P(z_j(t)))^3}{|z_j(t)|^p} = -\frac{d_*}{b^2} \cos(ps_j + d^*)$$

when  $p > n$ , and

$$(3.9) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re}(P(z_j(t)))^3}{|z_j|^n \log|z_j|} = (-1)^{j+1} \frac{2a_*^3}{b} \sin 3a^*$$

when  $p \leq n$ . By these (3.8) and (3.9) we can estimate the real parts of the functions  $P(z_j(t))$ . In fact it is clear that

$$\operatorname{Re}(P(z))^3 = (|P(z)|^2 - 4(\operatorname{Im} P(z))^2) \operatorname{Re}(P(z)).$$

Furthermore by virtue of (1.1)-(1.3),

$$\lim_{t \rightarrow \infty} \frac{|P(z_j)|^2 - 4(\operatorname{Im} P(z_j))^2}{|z_j(t)|^{2m}} = a_*^2 - 4a_*^2 \sin^2(ms_j + a^*)$$

for  $1 \leq j \leq 2n$ . Accordingly for each  $j$  such that  $\cos(ms_j + a^*) = 0$ ,

$$(3.10) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re} P(z_j(t))}{|z_j(t)|^{p-2m}} = \frac{d_*}{3a_*^2 b^2} \cos(ps_j + d^*)$$

when  $p > n$ , and

$$(3.11) \quad \lim_{t \rightarrow \infty} |z_j(t)|^{2m-n} \frac{\operatorname{Re} P(z_j(t))}{\log |z_j(t)|} = (-1)^j \frac{2a_*}{3b} \sin 3a^*$$

when  $p \leq n$ .

Using these (3.10) and (3.11), we next consider the real parts of the functions  $\overline{Q}(z_j)(P(z_j))^2$ . By the definition (3.3), it is evident that

$$(3.12) \quad b^2 \operatorname{Im}(\overline{Q}(z)(P(z))^3) = a_*^3 \sin 3a^* |Q(z)|^2 \operatorname{Re}(Q(z)) - \operatorname{Im}(g(z)\overline{Q}(z)).$$

By means of (1.1)-(1.3) and (3.4), we at once have

$$(3.13) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Im}(g(z_j(t))\overline{Q}(z_j(t)))}{|z_j(t)|^{n+p}} = (-1)^j b d_* \cos(ps_j + d^*)$$

for  $1 \leq j \leq 2n$ . Hence with the help of (1.1)-(1.3), (1.6), (3.12) and (3.13), for each  $j$  with  $1 \leq j \leq 2n$ ,

$$(3.14) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Im}((P(z_j))^3 \overline{Q}(z_j))}{|z_j(t)|^{n+p}} = (-1)^{j+1} \frac{d_*}{b} \cos(ps_j + d^*)$$

when  $p > n$ , and if  $p \leq n$ ,

$$(3.15) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Im}((P(z_j))^3 \overline{Q}(z_j))}{|z_j|^{2n} \log |z_j|} = -a_*^3 \sin 3a^*.$$

Furthermore by virtue of (1.1)-(1.3), we easily have

$$(3.16) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Im}((P(z_j))^2 \overline{Q(z_j)})}{|z_j(t)|^{2m+n}} = (-1)^{j+1} a_*^2 b$$

for each number  $j$  satisfying  $\cos(ms_j + a^*) = 0$ . Here let us note the elementary relation

$$\operatorname{Im}((P(z))^3 \overline{Q(z)}) = \operatorname{Re}(P(z)) \operatorname{Im}((P(z))^2 \overline{Q(z)}) + \operatorname{Im}(P(z)) \operatorname{Re}((P(z))^2 \overline{Q(z)}).$$

It then follows from (3.10), (3.11) and (3.14)-(3.16) that for each  $j$  with  $\cos(ms_j + a^*) = 0$ ,

$$(3.17) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re}((P(z_j))^2 \overline{Q(z_j)})}{|z_j(t)|^{n+p-m}} = (-1)^j \frac{2d_*}{3a_* b} \sin(ps_j - ms_j - a^* + d^*)$$

in the case  $p > n$ , and

$$(3.18) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re}((P(z_j))^2 \overline{Q(z_j)})}{|z_j|^{2n-m} \log|z_j|} = -\frac{a_*^2}{3} \cos(ms_j - 2a^*)$$

in the case  $p \leq n$ .

3.2. We are now in a position to show the following fact.

**Lemma 4.** *Let  $p$  be the integer defined by (3.4). Then for each number  $j$  satisfying  $\cos(ms_j + a^*) = 0$ ,*

$$(3.19) \quad \lim_{t \rightarrow \infty} |z_j(t)|^{3n-m-p} \operatorname{Re} \frac{w'_j(t)}{w_j(t)} = \frac{(p-2m)d_*}{3na_*^2 b^3} \cos(ps_j + d^*)$$

when  $p > 2m$ , and

$$(3.20) \quad \lim_{t \rightarrow \infty} |z_j(t)|^n \operatorname{Re} \frac{w'_j(t)}{w_j(t)} = \frac{1}{nb}$$

when  $p \leq 2m$ .

*Proof.* Differentiating (3.3), we obtain

$$g'(z) = 2a^3 Q(z)Q'(z) - 3b^2 (P(z))^2 P'(z).$$

Hence we have

$$(3.21) \quad \begin{aligned} & 2a_*^3 \sin 3a^* |Q(z)|^2 \operatorname{Im}(zP'(z)\overline{zQ'(z)}) \\ &= \operatorname{Re}(\overline{zg'(z)}zP'(z)Q(z)) + 3b^2 |zP'(z)|^2 \operatorname{Re}((P(z))^2 \overline{Q(z)}). \end{aligned}$$

By means of (1.1)-(1.3) and (3.4), it is clear that for every  $j$  with  $1 \leq j \leq 2n$ ,

$$(3.22) \quad \begin{aligned} & \lim_{t \rightarrow \infty} \frac{\operatorname{Re}(\overline{z_j g'(z_j)} z_j P'(z_j) Q(z_j))}{|z_j(t)|^{m+n+p}} \\ &= (-1)^j m p a_* b d_* \sin(ms_j - ps_j + a^* - d^*) \end{aligned}$$

when  $p \geq 1$ , and if  $p \leq 0$ ,

$$(3.23) \quad \operatorname{Re}(\overline{z_j g'(z_j)} z_j P'(z_j) Q(z_j)) = o(|z_j(t)|^{m+n})$$

as  $t$  grows to infinity. It therefore follows from (1.1)-(1.3), (3.17), (3.21) and (3.22) that if  $p > n$ ,

$$(3.24) \quad \begin{aligned} & \lim_{t \rightarrow \infty} \frac{\operatorname{Im}(z_j(t) P'(z_j(t)) \overline{z_j(t) Q'(z_j(t))})}{|z_j(t)|^{m-n+p}} \\ &= (-1)^j \frac{(p-2m)md_*}{2a_*^2 b} \cos(ps_j + d^*) \end{aligned}$$

for each  $j$  such that  $\cos(ms_j + a^*) = 0$ . Furthermore by making use of (1.1)-(1.3), (3.18) and (3.21)-(3.23), we deduce for every  $j$  satisfying  $\cos(ms_j + a^*) = 0$ ,

$$(3.25) \quad \operatorname{Im}(\overline{z_j(t) P'(z_j(t))} z_j(t) Q'(z_j(t))) = o(|z_j(t)|^n)$$

as  $t$  tends to infinity in the case  $p \leq n$ . If  $n < p < 2m$ , then  $m-n+p < n$ . If  $p = 2m$ , the limit value of (3.24) is clearly zero

and  $m-n+p=n$ . Hereby from (3.24), the estimation (3.25) remains valid for the case  $n < p \leq 2m$ . Consequently with the help of (1.10), the desired (3.19) and (3.20) follow from (3.24) and (3.25), respectively. The proof is now complete.

3.3. Let  $j$  be an integer between 2 and  $2n-1$  which satisfies

$$(3.26) \quad \cos(ms_j + a^*) = 0.$$

Then it is clear that

$$\sin 3a^* = \sin(ms_j + a^*),$$

and either  $\cos(ms_{j-1} + a^*)$  or  $\cos(ms_{j+1} + a^*)$  is positive by means of (3.2). Moreover  $\sin(ms_{j-1} + a^*)$  and  $\sin(ms_{j+1} + a^*)$  are equal to each other, and they are positive or negative according to whether  $\sin 3a^*$  is  $-1$  or  $1$ .

Suppose now that the real quantity  $p$  satisfies  $p \leq 2m$ . It then follows from (1.7) and (3.20) that

$$(3.27) \quad \lim_{t \rightarrow \infty} t \operatorname{Re} \frac{w_j'(t)}{w_j(t)} = \frac{1}{n}$$

for every  $j$  satisfying (3.26). For the imaginary part we have already obtained (1.12). Hence by using (1.7) again, we have

$$(3.28) \quad \lim_{t \rightarrow \infty} t^{\frac{1}{2}} \operatorname{Im} \frac{w_j'(t)}{w_j(t)} = \frac{2}{3} a_* b^{-\frac{1}{2}} \sin 3a^*$$

for each  $j$  with (3.26). Let  $l$  denote an integer between 1 and  $2n$  such that

$$(3.29) \quad \cos(ms_l + a^*) > 0, \quad \sin(ms_l + a^*) \sin 3a^* < 0.$$

Such an integer  $l$  surely exists by what is mentioned above. Then by virtue of (1.11) and (1.12),

$$(3.30) \quad \lim_{t \rightarrow \infty} |z_l(t)|^{n-m} \operatorname{Re} \frac{w_l'(t)}{w_l(t)} = \frac{2a_*}{3b} \cos(ms_l + a^*),$$



$$(3.31) \quad \lim_{t \rightarrow \infty} |z_l(t)|^{n-m} \operatorname{Im} \frac{w'_l(t)}{w_l(t)} = \frac{2a_*}{3b} \sin(ms_l + a_*).$$

Consequently by (1.7) and these (3.27)-(3.31), we at once obtain the desired (1.13)-(1.16). If  $\sin 3a_* = 1$ , then  $\alpha = 1$ ,  $\beta = \gamma = 1/3$  and  $R_k = 1/n$ . On the contrary if  $\sin 3a_* = -1$ , then  $\beta = 1$ ,  $\alpha = \gamma = 1/3$  and  $R_k = 1/n$ . We have therefore proved the main lemma when  $p \leq 2m$ .

Next let us consider the case  $p > 2m$ . In this case it is evident that  $4 < 3p/n < 6$ , so that

$$(3.32) \quad \sum \cos(ps_j + d_*) = 0,$$

where the sum ranges over all integers  $j$  between 1 and  $2n$  satisfying (3.26). Hence it is possible to find an integer  $j$  between 1 and  $2n$  such that

$$(3.33) \quad \cos(ps_j + d_*) > 0, \quad \cos(ms_j + a_*) = 0$$

except for the case where all the terms of (3.32) vanish. It thus follows from (3.19) and (3.33) that for such an integer  $j$ , the limit

$$\lim_{t \rightarrow \infty} |z_j(t)|^{3n-m-p} \operatorname{Re} \frac{w'_j(t)}{w_j(t)}$$

exists and is positive. Here notice that  $0 < 3n - m - p < n$  whenever  $p > 2m$ . Accordingly except when all the terms of (3.32) vanish, by the same reasoning just as above, we deduce the desired (1.13)-(1.16) with  $0 < \alpha < 1$ ,  $0 < \beta < 1$ ,  $\gamma = 1/3$ ,  $R_h$  and  $R_k$  being both positive.

3.4. Finally we must treat the above exceptional case, that is, the case in which

$$(3.34) \quad \cos(ps_j + d_*) = 0$$

for all numbers  $j$  satisfying (3.26). Evidently by the definition of  $s_j$ ,  $3(p-m)/n$  must be an integer in this case. Since  $2m < p$

$< 2n$ , it follows that  $3p=5n$ . Furthermore the arguments  $a^*$  and  $d^*$  must satisfy

$$(3.35) \quad \cos(a^* - d^*) = 0.$$

Again let  $j$  denote an integer between 1 and  $2n$  which satisfies (3.26). Then this number  $j$  satisfies (3.34). Hence we cannot make use of the estimation (3.19). We thus need a more precise estimate on the real part of the logarithmic derivative  $w'_j(t)/w_j(t)$ . To this end let us introduce the function

$$(3.36) \quad h(z) = g(z) - \frac{d}{ab} P(z)Q(z).$$

Since  $3m=2n$ , the condition  $3p=5n$  yields  $p=m+n$ . Hereby from (1.1) and (3.4), this function  $h(z)$  is regular or has a pole of order less than  $p$  at the point at infinity. It thus follows that with a suitable integer  $r$  less than  $p$ ,

$$(3.37) \quad h(z) = uz^r + O(z^{r-1})$$

near the point at infinity, where  $u$  is a nonzero constant. Here let us notice the conditions  $\cos 3a^*=0$  and (3.35). Then the constants  $a^3$  and  $d/ab$  are both purely imaginary. Hence by the definitions (3.3) and (3.36), it is clear that

$$\begin{aligned} \operatorname{Re} h(z) &= \operatorname{Re} g(z) + \frac{d_*}{a_*b} \sin(d^* - a^*) \operatorname{Im}(P(z)Q(z)) \\ (3.38) \quad &= -a_*^3 \sin 3a^* \operatorname{Im}(Q(z))^2 - b^2 \operatorname{Re}(P(z))^3 \\ &\quad + \frac{d_*}{a_*b} \sin(d^* - a^*) \operatorname{Im}(P(z)Q(z)). \end{aligned}$$

With the help of (1.1)-(1.3), (1.6), (3.37) and (3.38), we thus obtain that if  $r > n$ ,

$$(3.39) \quad \lim_{t \rightarrow \infty} \frac{\operatorname{Re} P(z_j(t))}{|z_j(t)|^{r-2m}} = \frac{u_*}{3a_*^2 b^2} \cos(rs_j + u^*)$$

and if  $r \leq n$ ,

$$(3.40) \quad \lim_{t \rightarrow \infty} |z_j(t)|^{2m-n} \frac{\operatorname{Re} P(z_j(t))}{\log |z_j(t)|} = (-1)^j \frac{2a_*}{3b} \sin 3a^*,$$

where  $u_* = |u|$  and  $u^* = \arg u$ . These are improvements of (3.10). Using the auxiliary function  $h(z)$  and the above (3.39) and (3.40) effectively, we can finally obtain the following result by the same method as before.

**Lemma 5.** *Let  $j$  be an arbitrary integer between 1 and  $2n$  such that  $\cos(ms_j + a^*) = 0$ . If the integer  $r$  defined with (3.37) is greater than  $2m$ ,*

$$\lim_{t \rightarrow \infty} |z_j(t)|^{2m+n-r} \operatorname{Re} \frac{w_j'(t)}{w_j(t)} = \frac{(r-2m)u_*}{3n a_*^2 b^3} \cos(rs_j + u^*).$$

If  $r \leq 2m$ , then

$$\lim_{t \rightarrow \infty} |z_j(t)|^n \operatorname{Re} \frac{w_j'(t)}{w_j(t)} = \frac{1}{nb}.$$

By virtue of this Lemma 5, if  $r \leq 2m$ , we at once have the desired results (1.13)-(1.16) with  $\alpha = 1$  and  $R_h = 1/n$ , or  $\beta = 1$  and  $R_k = 1/n$ .

Suppose that  $r > 2m$ . Then  $4 = 6m/n < 3r/n < 3p/n = 5$ . In particular  $3r/n$  is not integer. Hence there surely exists an integer  $j$  among 1 to  $2n$  such that

$$\cos(ms_j + a^*) = 0, \quad \cos(rs_j + u^*) > 0.$$

Furthermore notice that  $2n/3 < 2m + n - r < n$ . Consequently by the same reasoning as before, we deduce (1.13)-(1.16) with  $2/3 < \alpha < 1$  and  $\beta = \gamma = 1/3$ , or  $2/3 < \beta < 1$  and  $\alpha = \gamma = 1/3$ . Accordingly we have completely proved the main lemma in the

case  $3m=2n$ . The main lemma is now established.

**4. The proof of Theorem.** Let  $A(z)$ ,  $B(z)$ ,  $P(z)$  and  $Q(z)$  be nonconstant regular functions in the annulus  $0 < |z| < +\infty$ . Suppose that these four functions satisfy

$$(4.1) \quad P(z) - Q(z) = A(ze^{P(z)}) - B(ze^{Q(z)})$$

in this annulus. Suppose further that the functions  $P(z)$  and  $Q(z)$  both have poles at the point at infinity. We denote the orders of the poles of  $P(z)$  and  $Q(z)$  by  $m$  and  $n$ , respectively. Then

$$(4.2) \quad P(z) = az^m + O(z^{m-1}), \quad Q(z) = bz^n + O(z^{n-1})$$

near the point at infinity, where  $a$  and  $b$  are nonzero constants.

Our aim is to show that  $m=n$ . On the contrary we assume henceforth that the orders  $m$  and  $n$  are distinct. Then by the symmetrical property of the functional equation (4.1), we may assume that  $1 \leq m < n$ . Furthermore we may assume without loss of generality that the constant  $b$  of (4.2) is real positive.

Let  $z_j(t)$  be the  $2n$  functions of Lemma 1, and let  $w_j(t)$  be the  $2n$  functions defined by

$$(4.3) \quad w_j(t) = z_j(t) \exp(P(z_j(t)))$$

for  $t \geq 0$  ( $j=1, 2, \dots, 2n$ ). Put these functions  $z_j(t)$  into the functional equation (4.1). Then by the property (1.4) and the definition (4.3),

$$A(w_j(t)) - P(z_j(t)) + Q(z_j(t))$$

must be bounded for  $t \geq 0$  ( $j=1, 2, \dots, 2n$ ). It thus follows from (1.2) and (4.2) that

$$\lim_{t \rightarrow \infty} \frac{A(w_j(t))}{(z_j(t))^n} = -b,$$

so that by means of (1.3) and (1.7),

$$(4.4) \quad \lim_{t \rightarrow \infty} \frac{A(w_j(t))}{t} = (-1)^j i$$

for each  $j$  with  $1 \leq j \leq 2n$ . On the other hand by virtue of the main lemma, we can choose two functions  $w_h(t)$  and  $w_k(t)$  among the  $2n$  functions  $w_j(t)$  which satisfy (1.13)-(1.16). The real quantities  $R_h$  and  $R_k$  are both positive or negative.

4.1. Suppose first that the real constants  $R_h$  and  $R_k$  are both positive. Then by means of (1.13)-(1.16), we can take a positive number  $t^*$  such that

$$(4.5) \quad t^\alpha \operatorname{Re}(w_h'(t)/w_h(t)) > R_h/2 > 0,$$

$$(4.6) \quad t^\beta \operatorname{Re}(w_k'(t)/w_k(t)) > R_k/2 > 0,$$

$$(4.7) \quad t^r \operatorname{Im}(w_h'(t)/w_h(t)) > I_h/2 > 0,$$

$$(4.8) \quad t^r \operatorname{Im}(w_k'(t)/w_k(t)) < I_k/2 < 0$$

for all real values of  $t$  with  $t \geq t^*$ . Since the quantities  $\alpha$  and  $\beta$  are real numbers of the interval  $0 < x \leq 1$ , it follows from (4.5) and (4.6) that the functions  $|w_h(t)|$  and  $|w_k(t)|$  are strictly increasing for  $t \geq t^*$  and become infinite when  $t$  tends to infinity. More precisely by making use of l'Hospital's rule, we deduce from (1.13) that

$$(4.9) \quad \lim_{t \rightarrow \infty} t^{\alpha-1} \log |w_h(t)| = R_h/(1-\alpha)$$

when  $0 < \alpha < 1$ , and if  $\alpha = 1$ , then

$$(4.10) \quad \lim_{t \rightarrow \infty} \frac{\log |w_h(t)|}{\log t} = R_h = \frac{1}{n}.$$

Similarly by means of (1.14),

$$(4.11) \quad \lim_{t \rightarrow \infty} t^{\beta-1} \log |w_k(t)| = R_k/(1-\beta)$$

when  $0 < \beta < 1$ , and if  $\beta = 1$ , then  $R_k = 1/n$  and

$$(4.12) \quad \lim_{t \rightarrow \infty} \frac{\log |w_k(t)|}{\log t} = \frac{1}{n}.$$

Let  $x^*$  be the maximum of  $|w_h(t^*)|$  and  $|w_k(t^*)|$ . Then we can define the inverses of the functions  $x=|w_h(t)|$  and  $x=|w_k(t)|$  on the interval  $x \geq x^*$ . We denote these inverse functions of  $|w_h(t)|$  and  $|w_k(t)|$  by  $u(x)$  and  $v(x)$ , respectively. Obviously these functions  $u(x)$  and  $v(x)$  increase monotonically for  $x \geq x^*$ , and they converge to infinity as  $x$  goes to infinity.

Next let us consider the functions

$$(4.13) \quad I(x) = \arg w_h(u(x)), \quad J(x) = \arg w_k(v(x))$$

for  $x \geq x^*$ , where the branches are chosen so that  $0 \leq I(x^*) < 2\pi$  and  $0 \leq J(x^*) < 2\pi$ . Then with the help of (4.7), the function  $I(x)$  increases steadily for  $x \geq x^*$ . Furthermore since  $0 < \gamma < 1$ ,  $I(x)$  converges to infinity as  $x$  tends to infinity. Similarly by means of (4.8), the function  $J(x)$  decreases steadily for  $x \geq x^*$  and becomes negatively infinite as  $x$  goes to infinity. Accordingly the difference  $I(x) - J(x)$  is strictly increasing and unbounded for  $x \geq x^*$ . Hence for each natural number  $j$ , there exists a uniquely determined number  $x_j$  such that  $x_j \geq x^*$  and

$$(4.14) \quad I(x_j) - J(x_j) = 2\pi j.$$

Evidently  $x_j < x_{j+1}$  for  $j \geq 1$ , and  $x_j$  goes to infinity with  $j$ . On the other hand it is indeed clear from definition that

$$(4.15) \quad \begin{aligned} w_h(u(x)) &= x \exp(iI(x)), \\ w_k(v(x)) &= x \exp(iJ(x)) \end{aligned}$$

for  $x \geq x^*$ . In particular the above relation (4.14) yields

$$w_h(u(x_j)) = w_k(v(x_j))$$

for  $j \geq 1$ . It therefore follows that the arcs

$$(4.16) \quad C_j^+ = \{w_h(t) : u(x_j) \leq t \leq u(x_{j+1})\},$$

$$(4.17) \quad C_j^- = \{w_k(t) : v(x_j) \leq t \leq v(x_{j+1})\}$$

have the same initial point  $w_h(u(x_j))$  and the same final point  $w_h(u(x_{j+1}))$ . In view of (4.15), the arc  $C_j^+$  winds around the origin in the positive direction while the arc  $C_j^-$  does in the negative direction when the real parameter  $t$  increases. The variations of the argument along  $C_j^+$  and  $C_j^-$  are clearly  $I(x_{j+1}) - I(x_j)$  and  $J(x_{j+1}) - J(x_j)$ , respectively. Consequently the curve  $C_j = C_j^+ - C_j^-$  which consists of the arc  $C_j^+$  and the arc  $C_j^-$  in the opposite direction must be simple closed, and it lies entirely in the annulus  $x_j \leq |z| \leq x_{j+1}$ . Furthermore according to (4.14), the winding number of  $C_j$  with respect to the origin is equal to one.

Here recall (4.9)-(4.12). Then it follows that

$$(4.18) \quad \lim_{t \rightarrow \infty} \frac{t}{|w_h(t)|^{n+1}} = 0, \quad \lim_{t \rightarrow \infty} \frac{t}{|w_k(t)|^{n+1}} = 0.$$

Hence combining this (4.18) with (4.4), we deduce that

$$\lim_{t \rightarrow \infty} \frac{A(w_h(t))}{(w_h(t))^{n+1}} = 0, \quad \lim_{t \rightarrow \infty} \frac{A(w_k(t))}{(w_k(t))^{n+1}} = 0.$$

Therefore by the definitions (4.16) and (4.17), the function  $A(z)/z^{n+1}$  must be uniformly bounded on the simple closed curves  $C_j$ . We can thus assert by the nature of the curves  $C_j$  that the function  $A(z)/z^{n+1}$  is bounded in a neighborhood of the point at infinity, so that it must be regular there. Hereby with a suitable integer  $s$ ,

$$\lim_{t \rightarrow \infty} \frac{A(w_h(t))}{(w_h(t))^s} = C,$$

where  $C$  is a nonzero constant. According to (4.4) again, we hence obtain

$$\lim_{t \rightarrow \infty} \frac{t}{(w_h(t))^s} = (-1)^{h+1} iC.$$

This implies that  $s \neq 0$ , so that the argument of  $w_h(t)$  must be bounded for  $t \geq t^*$ . This is a contradiction. Consequently if  $R_h$  and  $R_k$  are both positive, we reach a contradiction.

4.2. It therefore remains to consider the case when the real constants  $R_h$  and  $R_k$  are both negative. In this case with the help of (1.13)-(1.16), we can choose a positive number  $t_*$  such that

$$t^\alpha \operatorname{Re}(w_h'(t)/w_h(t)) < R_h/2 < 0,$$

$$t^\beta \operatorname{Re}(w_k'(t)/w_k(t)) < R_k/2 < 0$$

and the inequalities (4.7) and (4.8) hold for all real values of  $t$  with  $t \geq t_*$ . Then  $|w_h(t)|$  and  $|w_k(t)|$  both strictly decrease for  $t \geq t_*$  and converge to zero as  $t$  tends to infinity. Hence we can consider the inverses of the functions  $x = |w_h(t)|$  and  $x = |w_k(t)|$  on the interval  $0 < x \leq x_*$ , where  $x_*$  is a sufficiently small positive constant. As before we denote these inverse functions of  $|w_h(t)|$  and  $|w_k(t)|$  by  $u(x)$  and  $v(x)$ , respectively. In this case  $u(x)$  and  $v(x)$  both decrease monotonically for  $0 < x \leq x_*$ , and they become positively infinite as  $x$  goes to zero. The arguments of  $w_h(t)$  and  $w_k(t)$  are strictly increasing and strictly decreasing for  $t \geq t_*$ , respectively. Furthermore they are unbounded there. According to these facts the function

$$S(x) = \arg w_h(u(x)) - \arg w_k(v(x)),$$

where the branch is taken so that  $0 \leq S(x_*) < 2\pi$ , must strictly increase indefinitely as  $x$  decreases from  $x_*$  to 0. Hereby for each natural number  $j$ , there exists a uniquely determined number  $y_j$  such that  $0 < y_j \leq x_*$  and  $S(y_j) = 2\pi j$ . Obviously the sequence  $\{y_j\}$  is steadily decreasing and converges to zero. Using this sequence  $\{y_j\}$  we define the arcs

$$L_j^+ = \{w_h(t) : u(y_j) \leq t \leq u(y_{j+1})\},$$



$$L_j^- = \{w_k(t) : v(y_j) \leq t \leq v(y_{j+1})\}$$

for  $j \geq 1$ . Then each arc  $L_j^+$  lies in the annulus  $y_{j+1} \leq |z| \leq y_j$  and winds around the origin in the positive direction as the parameter  $t$  moves from  $u(y_j)$  to  $u(y_{j+1})$ . Similarly every arc  $L_j^-$  also lies in that annulus entirely and winds around the origin in the negative direction when  $t$  varies from  $v(y_j)$  to  $v(y_{j+1})$ . Furthermore the arcs  $L_j^+$  and  $L_j^-$  both join the point  $w_h(u(y_j))$  to the point  $w_h(u(y_{j+1}))$ . Hence because of the relation  $S(y_{j+1}) - S(y_j) = 2\pi$ , the curves  $L_j = L_j^+ - L_j^-$  are all simple closed and wind around the origin exactly once.

Here let us remark that neither  $\alpha$  nor  $\beta$  is equal to one in this case. Then (4.9) and (4.11) hold. Hence it is clear that

$$\lim_{t \rightarrow \infty} t w_h(t) = 0, \quad \lim_{t \rightarrow \infty} t w_k(t) = 0.$$

We thus deduce from (4.4) that

$$(4.19) \quad \begin{aligned} \lim_{t \rightarrow \infty} w_h(t) A(w_h(t)) &= 0, \\ \lim_{t \rightarrow \infty} w_k(t) A(w_k(t)) &= 0. \end{aligned}$$

By the manner in which the curves  $L_j$  have been constructed, this (4.19) means that the function  $zA(z)$  is uniformly bounded on the curves  $L_j$ . Therefore by the maximum principle, this function  $zA(z)$  must be bounded in a neighborhood of the origin because all the simple closed curves  $L_j$  wind around the origin and converge to it. Accordingly with the help of (4.19), the function  $A(z)$  must be regular at the origin, so that  $A(w_k(t))$  converges to some finite value as  $t$  goes to infinity. However this clearly contradicts (4.4). Consequently if the real constants  $R_h$  and  $R_k$  are both negative, we also obtain a contradiction. This completes the proof.

### References

- [1] L. Ahlfors, Complex analysis, McGraw-Hill, New York, 1953.
- [2] T. Kobayashi, Factorization of certain entire functions, Chiba Keiai University, 26 (1984), 135-171.