Remarks on factorization of entire functions

By Tadashi Kobayashi

If a meromorphic function is representable as $f_1(g_1(z))$ and $f_2(g_2(z))$, where $f_i(z)$ and $g_i(z)$ are meromorphic, and if with a suitable linear transformation $l(z)$, $f_2(z)=f_1(l(z))$ and $g_2(z)=l^{-1}(g_1(z))$ hold, then the two representations or factorizations are called to be equivalent. A meromorphic function $F(z)$ is said to be uniquely factorizable if every nontrivial factorization of $F(z)$ is equivalent to one another. When factors are restricted to entire functions, it is called to be a factorization in entire sense. In our previous paper [4] we have proved the following fact.

Let $A(z)$ and $B(z)$ be transcendental entire functions such that the composite function $A(B(z))$ is uniquely factorizable in entire sense. If this function $A(B(z))$ is not uniquely factorizable relative to the family of meromorphic functions, then $A(z)$ or $A(B(z))$ must be a periodic function.

It is natural to ask whether there exists an entire function which satisfies the assumption of the above theorem. Concerning this question we shall consider the factorization of certain entire functions.

1. First of all we shall prove the following theorem.

Theorem 1. *Let $A(z)$ be $\exp(-z)+\exp(2z)$ and let $B(z)$ be ze^z. Then the composite function $A(B(z))$ is uniquely factorizable relative to the family of entire functions.*

Evidently this composite function $A(B(z))$ is not uniquely factorizable. In fact $A(B(z))=A^*(B^*(z))$, where $A^*(z)=z^{-1}+z^2$ and

$B^*(z)=\exp(ze^z)$.

For the sake of simplicity we denote the composite function $A(B(z))$ with $H(z)$. Now suppose that the function $H(z)$ has another factorization $f(g(z))$ with nonlinear entire functions $f(z)$ and $g(z)$. Then

$$H(z)=A(B(z))=f(g(z)). \tag{1.1}$$

Differentiation of (1.1) yields

$$\begin{aligned} H'(z) &= f'(g(z))g'(z) \\ &= (1+z)(2\exp(3B(z))-1)\exp(z-B(z)). \end{aligned} \tag{1.2}$$

Therefore, except for the point $z=-1$, all the zeros of $H'(z)$ satisfy $2\exp(3B(z))=1$, hence $4(H(z))^3=27$. Since $H'(-1)=0$, either $f'(g(-1))=0$ or $g'(-1)=0$. Hereafter our consideration is divided into two parts.

Part I. Let us consider the case $f'(g(-1))=0$. Suppose that there exists a point v such that $g(v)=g(-1)$. Then $H'(v)=0$ by assumption and (1.2). Hence $4(H(v))^3=27$ unless $v=-1$. On the other hand it is clear from (1.1) that $H(v)=H(-1)$. Let x^* be the real root of the equation $2\exp(3z)=1$. Then $B(-1)<x^*<0$ and $A(x)>A(x^*)$ for all real values of x other than x^*. Hence $4(H(-1))^3\neq 27$. Consequently the point v must be -1. Thereby the function $g(z)$ takes the value $g(-1)$ only at the point -1. Further $H''(-1)\neq 0$, so that $g'(-1)\neq 0$ by assumption and (1.2). It thus follows that the right factor $g(z)$ must be transcendental and has the value $g(-1)$ as an asymptotic value. Therefore from (1.1), the function $H(z)$ has the value $H(-1)$ as an asymptotic value. The value $H(-1)$ is not equal to 2. In fact, since $\log 3/2>1/e$, by setting

$$u=\log(5^{1/2}-1)-\log 2,$$

we find that $u<B(-1)=-1/e<x^*$. Furthermore the real function $A(x)$ is strictly decreasing for $x\leqq x^*$. Hence $A(u)>A(B(-1))=H(-1)$. Evidently $A(u)=2$, so that $H(-1)\neq 2$.

Now let us consider the algebraic equation $z^{-1}+z^2=H(-1)$. Since

$4(H(-1))^3 \neq 27$, this equation has exactly three distinct solutions. Of course, none of the three roots is zero. We denote these roots by z_1, z_2 and z_3. It thus follows that

$$z^{-1}+z^2-H(-1)=z^{-1}(z-z_1)(z-z_2)(z-z_3). \tag{1.3}$$

Here let us notice the representation

$$H(z)=\exp(-B(z))+\exp(2B(z)).$$

Then with the help of (1.3), we at once obtain

$$H(z)-H(-1)=(B^*(z))^{-1}(B^*(z)-z_1)(B^*(z)-z_2)(B^*(z)-z_3),$$

where $B^*(z)=\exp(B(z))$. Since $H(z)$ converges to the value $H(-1)$ when z tends to the infinity along some asymptotic curve, the function $B^*(z)$ also approaches one of the values z_j as z tends to the infinity along the asymptotic curve. Hereby one of the three values z_j is an asymptotic value of the function $B^*(z)$. Accordingly one of the values $\log z_j$ is an asymptotic value of the function $B(z)$. On the other hand it is easily verified that the function $B(z)$ has no finite asymptotic values other than zero. It thus follows that one of the three solutions z_j must be one. Hence $H(-1)=2$ by (1.3). This is a contradiction. Consequently the case $f'(g(-1))=0$ never occurs.

Part II. We can therefore assume that $f'(g(-1))\neq 0$. For each real positive number x, let us consider the real function

$$L(x,y)=y+\arctan y/x \tag{1.4}$$

for $y\geqq 0$, where the branch is taken so that $L(x,0)=0$. Then it is clear that

$$y\leqq L(x,y)\leqq y+\pi/2 \tag{1.5}$$

for $y\geqq 0$ and $L(x,y)$ is a strictly increasing function of y. Hence for each positive x, we can determine the real positive number $s(x)$ uniquely so that

$$L(x,s(x))=\pi. \tag{1.6}$$

In view of (1.5), $\pi/2<s(x)<\pi$ for $x>0$. Since the inverse of tangent is increasing, $L(x, s(x))>L(t, s(x))$ for any real positive values of x and t with $x<t$. It thus follows from the definition (1.6) that $s(x)<s(t)$ for $0<x<t$, so that the quantity $s(x)$ increases steadily with x. Using this property we can easily assert from the construction of $s(x)$ that the function $s(x)$ is continuous for $x>0$. Furthermore it follows from (1.4) and (1.6) that $s(x)$ converges to $\pi/2$ as x tends to positively zero.

Next for each real x between -1 and 0, we consider the function $L(x,y)$ defined with (1.4). For this case the branch of the inverse of tangent is chosen so that $L(x, 0)=\pi$. Then

$$y+\pi/2\leqq L(x,y)\leqq y+\pi,$$

$$\frac{d}{dy}L(x,y)=1+x(x^2+y^2)^{-1}$$

for $y\geqq 0$. Hence $L(x,y)$ steadily decreases for $0\leqq y\leqq(-x-x^2)^{1/2}$ and steadily increases for $y\geqq(-x-x^2)^{1/2}$. We can thus determine the real positive number $s(x)$ uniquely which satisfies (1.6). It is clear that

$$0<(-x-x^2)^{1/2}<s(x)\leqq\pi/2 \tag{1.7}$$

for $-1<x<0$. Let u and v be arbitrary real numbers with $-1<u<v<0$. Then $L(u, s(u))>L(v, s(u))$ by the same reasoning as before. Hence $s(u)<s(v)$ by the construction (1.6). This means that the function $s(x)$ also increases strictly with x, so that it is continuous for $-1<x<0$. Since $s(x)/x$ tends to negatively infinite as x converges to negatively zero, $s(x)$ approaches $\pi/2$ from below when x tends to negatively zero. On the other hand from (1.7), $s(x)$ converges to some nonnegative value s when x tends to -1. In veiw of (1.4) and (1.6), this limit s must satisfy $\tan s=s$, so that $s=0$. Hereby $s(x)$ converges to zero when x tends to -1. Consequently by setting $s(-1)=0$ and $s(0)=\pi/2$, the function $s(x)$ is strictly increasing and continuous for $x\geqq -1$. Furthermore $s(x)$ is differentiable for $x>-1$. Indeed by means of (1.6), $\tan s(x)=-s(x)/x$ for $x\geqq -1$ with $x\neq 0$. Hence

$$(uv+s(u)s(v))\tan(s(u)-s(v))=us(v)-vs(u)$$

for any real values of u and v in the interval $x \geqq -1$. It thus follows from the continuity of $s(x)$ that

$$\frac{us(v)-vs(u)}{s(u)-s(v)} \longrightarrow u^2+(s(u))^2$$

as v tends to u. By this fact we can assert the differentiability of $s(x)$ and obtain

$$s'(x)=\frac{s(x)}{x^2+x+(s(x))^2} \tag{1.8}$$

for $x>-1$.

We now set the simply connected region

$$G=\{x+iy: x>-1,\ -s(x)<y<s(x)\}.$$

The boundary of this region G is the simple curve defined with

$$z(x)=x+is(x),\quad \overline{z(x)}=x-is(x)$$

for $x \geqq -1$. From now on let us investigate the function $B(z)$ on this region G. For arbitrary real values of x and y with $x \geqq -1$ and $y \geqq 0$, it follows from the definition (1.4) that

$$\begin{aligned} B(x+iy) &= (x+iy)\exp(x+iy) \\ &= |x+iy| \exp(x+iL(x,y)), \end{aligned}$$

here we set for convenience that $L(0,y)=y+\pi/2$. Hence the argument of $B(x+iy)$ is equal to $L(x,y)$. Hereby for an arbitrarily fixed real positive x, the argument of $B(x+iy)$ increases steadily from 0 to π when y varies from 0 to $s(x)$. Obviously $|B(x+iy)| \geqq xe^x$, so that $B(z)$ converges uniformly to infinity as z tends to infinity from the inside of this region G. On the boundary of G,

$$B(z(x))=B(\overline{z(x)})=-|x+is(x)|e^x$$

for $x \geqq -1$. Since

$$x+s(x)s'(x)=\frac{(1+x)(x^2+(s(x))^2)}{x^2+x+(s(x))^2}>0$$

by means of (1.7) and (1.8), $x^2+(s(x))^2$ steadily increases with x. Therefore the boundary values $B(z(x))=B(\overline{z(x)})$ are always real negative and decrease strictly from $B(-1)=-1/e$ to negatively infinite as x grows from -1 to positively infinite. By these facts, the function $B(z)$ maps the region G conformally onto the slit region S whose slit is the half line $x \leqq -1/e$ of the real axis. We can thus consider the inverse function of $B(z)$ which effects a conformal mapping of the slit region S onto the region G. With $C(z)$ we denote this inverse function. Here it should be imagined that the slit of the region S has two edges. Then the function $C(z)$ maps the upper and the lower edges onto the curves $z=z(x)$ $(x \geqq -1)$ and $z=\overline{z(x)}$ $(x \geqq -1)$, respectively.

By means of these simple preliminaries we can prove our assertion. Let us recall the functional equation (1.1). Then since $B(z(x))=B(\overline{z(x)})$ for $x \geqq -1$, it follows that

$$(1.9) \qquad f(g(z(x)))=f(g(\overline{z(x)}))$$

for all real values of x with $x \geqq -1$. In view of the condition $f'(g(-1)) \neq 0$, the factor $f(z)$ is univalent in a neighborhood of the point $g(-1)$. Of course, $z(-1)=\overline{z(-1)}=-1$. Hereby from (1.9), it is possible to find a real positive number ε so that

$$(1.10) \qquad g(z(x))=g(\overline{z(x)})$$

for real values of x with $-1 \leqq x < -1+\varepsilon$. Assume for a moment that (1.10) holds at some real t with $t>-1$. Since $B(z(t))$ is real and less than $-1/e$, $3B(z(t))<-\log 2$. Of course, $z(t) \neq -1$. It thus follows from (1.2) that $H'(z(t))$ is different from zero, so that $f'(g(z(t))) \neq 0$. Consequently the left factor $f(z)$ is also univalent in a neighborhood of the point $g(z(t))$. Hence from (1.9) again, we can take a positive number δ such that the above (1.10) holds for all real values of x with $|x-t|<\delta$. Accordingly the equation (1.10) holds for all real values of x with $x \geqq -1$, because the set $x \geqq -1$ is connected. With the help of this relation (1.10), the composite function $g(C(z))$ must be entire. In fact this function $g(C(z))$ is defined and regular in the slit region S and is continuous on the two edges of the slit. Let w be a point of the slit. Then $w=B(z(x))=B(\overline{z(x)})$ with the uniquely determined real number x. We may consider by construction that the point $w=B(z(x))$ is a point of the upper edge while the point $w=B(\overline{z(x)})$ is a point of the lower edge. Then the boundary values of $g(C(z))$ at the points $w=B(z(x))$ and $w=B(\overline{z(x)})$ agree with $g(z(x))$ and $g(\overline{z(x)})$, respectively. Therefore by virtue of (1.10), the function $g(C(z))$ becomes single valued on the slit of S. It hence follows from a result of Painleve that the function $g(C(z))$ is regular at each point of the slit, so that this function is regular in the whole finite plane.

Let us put the inverse function $C(z)$ into the functional equation (1.1). Then we have the relation

$$A(z)=f(g(C(z))) \tag{1.11}$$

on the slit region S. Since the composite function $g(C(z))$ can be extended to an entire function, we can regard this relation (1.11) as a factorization of the function $A(z)$. However by a result due to Gross [2], the function $A(z)$ has no nontrivial factorizations in entire sense. Hence $f(z)$ or $g(C(z))$ must be a linear function. Consequently the entire function $g(C(z))$ is linear, so that the right factor $g(z)$ can be written in the form

$$g(z)=aB(z)+b$$

with suitable constants a and b. We thus complete the proof.

2. An entire function $F(z)$ is said to be prime if every factorization $F(z)=f(g(z))$ implies that either the left factor $f(z)$ or the right factor $g(z)$ is linear.

Theorem 2. *Let $S(z)$ be a prime entire function in entire sense which has only simple zeros. Let p be a prime integer with*

$p \geqq 3$. *Suppose that the entire function* $(S(z))^p$ *is not uniquely factorizable relative to the family of entire functions. Then the prime function* $S(z)$ *can be written in the form*

$$S(az+b)=z\,E(z^p),$$

where a *and* b *are constants with* $a \neq 0$, *and* $E(z)$ *is an entire function.*

Proof. Let $f(z)$ and $g(z)$ be nonlinear entire functions such that

$$(S(z))^p=f(g(z)). \tag{2.1}$$

In general a prime entire function has at least one zero-point. Hence $S(z)$ has at least one zero, so does the left factor $f(z)$ by the relation (2.1). Let w be a zero of $f(z)$. Suppose for a moment that the right factor $g(z)$ fails to take the value w. Then with a suitable nonconstant entire function $h(z)$,

$$g(z)-w=\exp(h(z)). \tag{2.2}$$

Furthermore $f(z)$ has other zeros. Let u be another zero of $f(z)$. Then the equation $g(z)=u$ has infinitely many solutions, and by virtue of the well known Nevanlinna's ramification relation [5], all but a finite number of the solutions are simple. On the other hand all the zeros of $(S(z))^p$ have multiplicity p. Hence the multiplicity of the zero u is exactly equal to p. This means that except for the zero w, all the zeros of $f(z)$ have multiplicity p. Hereby with a suitable nonconstant entire function $K(z)$, we can express the function $f(z)$ as

$$f(z)=(z-w)^n(K(z))^p, \tag{2.3}$$

where n is a natural number. Combining (2.2) and (2.3), we at once obtain

$$f(g(z))=(K(g(z)))^p \exp(nh(z)).$$

It thus follows from (2.1) that

$$S(z)=K^*(g(z)) \exp(nh(z)/p), \tag{2.4}$$

where $K^*(z)$ is a nonconstant entire function. Here let us set

$$f^*(z)=z^n K^*(w+z^p), \quad g^*(z)=\exp(h(z)/p).$$

Then it is clear from (2.2) and (2.4) that $S(z)=f^*(g^*(z))$. However neither $f^*(z)$ nor $g^*(z)$ is linear. This contradicts the primeness of $S(z)$.

Again let w be a zero of the left factor $f(z)$. By what we have shown just above, the equation $g(z)=w$ has solutions. Suppose that the zero w is not simple. Then all the roots of $g(z)=w$ are simple, because the integer p is prime. Hence by the same reasoning as before, the multiplicity of this zero w is equal to p. On the contrary if the zero w is simple, then all the solutions of $g(z)=w$ have multiplicity p. Therefore since $p\geqq 3$, by virtue of the ramification relation, $f(z)$ has at most one simple zero.

We now consider the case where $f(z)$ has no simple zeros. In this case all the zeros of $f(z)$ have multiplicity p. Hence there exists an entire function $f_*(z)$ such that $f(z)=(f_*(z))^p$. It thus follows from the functional equation (2.1) that $S(z)=\omega f_*(g(z))$ with a suitable pth root of unity ω. By assumption the right factor $g(z)$ is not a linear function. Hence by the primeness of $S(z)$, $f_*(z)$ must be linear. Consequently the two factorizations $(S(z))^p$ and $f(g(z))$ are equivalent. We may thus reject this case.

Hereby the left factor $f(z)$ has exactly one simple zero and all other zeros have multiplicity just p. We denote the simple zero by w. Then with a suitable nonconstant entire function $D(z)$, we can express $f(z)$ as

$$f(z)=(z-w)\,(D(z))^p. \tag{2.5}$$

Furthermore by what we have proved above, for this simple zero w, all the roots of the equation $g(z)=w$ have multiplicity p. Hence with a suitable entire function $G(z)$, the right factor $g(z)$ can be written in the form

$$g(z)=w+(G(z))^p. \tag{2.6}$$

Inserting this expression (2.6) into (2.5), we thus obtain

$$f(g(z))=(G(z)D(g(z)))^p,$$

so that $S(z)=G(z)D(g(z))$ by means of (2.1). By setting $E(z)=D(z+w)$ and $E^*(z)=zE(z^p)$, we finally have the factorization $S(z)=E^*(G(z))$. Since $p\geqq 3$ and $E(z)$ is nonconstant entire, $E^*(z)$ does not reduce to a linear function. Hence $G(z)$ must be linear. It therefore follows that

$$S(z)=(a'z+b')E((a'z+b')^p)$$

with constants a' and b', which is precisely what we wanted to prove. This completes the proof.

Let us consider the entire function $S(z)=z\exp(z^p)$, where p is a prime integer greater than two. Then this function $S(z)$ is prime and has only simple zeros. By setting $f(z)=z\exp(pz)$ and $g(z)=z^p$, we at once obtain $(S(z))^p=f(g(z))$. Of course, these two factorizations are not equivalent. Hereby the function $S(z)$ is an example of Theorem 2.

From Theorem 2 we immediately obtain the following.

Corollary. *Let $S(z)$ be an entire function which is prime in entire sense. Assume that all the zeros of $S(z)$ are simple and $S''(z)\neq 0$ for all the zeros. Then for each prime integer p greater than two, the composite function $(S(z))^p$ is uniquely factorizable in entire sense.*

Proof. Suppose that the composite function $(S(z))^p$ is not uniquely factorizable in entire sense. Then by means of Theorem 2, with some constants a and b,

$$S(az+b)=zE(z^p), \tag{2.7}$$

where $E(z)$ is an entire function. Clearly the quantity b satisfies $S(b)=0$. Hence b is a zero of $S(z)$, so that $S''(b)$ is not zero by assumption. However by differentiating (2.7) twice, we obtain

$$a^2S''(az+b)=p(p+1)z^{p-1}E'(z^p)+p^2z^{2p-1}E''(z^p).$$

Consequently $S''(b)=0$. This is impossible. The proof of Corollary is complete.

By making use of this Corollary we can present many entire functions which are uniquely factorizable in entire sense but not uniquely factorizable relative to the family of meromorphic functions. For instance let us set $S(z)=\exp(-mz)+\exp(nz)$, where m and n are distinct natural numbers. Then by the result due to Gross, this periodic entire function $S(z)$ is prime in entire sense. Further all the zeros of $S(z)$ are surely simple and satisfy

$$S''(z)=(m^2-n^2)\exp(-mz).$$

Therefore for every prime integer p greater than two, $(S(z))^p$ is uniquely factorizable in entire sense.

3. In this final section we shall prove the following result which is a refinement of the Gross' theorem referred to in the end of the previous section.

Theorem 3. *Let* $S(z)$ *be* $\exp(Az)+\exp(Bz+C)$, *where* A, B *and* C *are constants with* $AB\neq 0$. *Then the function* $S(z)$ *is prime in entire sense if and only if* $A+B\neq 0$ *and the ratio* A/B *is not any real positive rational number.*

Necessity is clear. If A is equal to $-B$, then

$$S(z)=2\exp(C/2)\cos(iAz-iC/2).$$

Evidently the cosine function is not prime. If $nA=mB$ with two natural numbers m and n, then by setting

$$f(z)=z^m+\exp(C)z^n, \quad g(z)=\exp(Az/m),$$

we at once have the factorization $S(z)=f(g(z))$.

For sufficiency we need two lemmas.

Lemma 1. *Let $g(z)$ be an entire function having only real zeros and real ones. Then either $g(z)$ takes only real values on the real axis or $g(z)$ is periodic with real period. Furthermore in the former case, $g(z)$ has no finite asymptotic values.*

Lemma 2. *Let $P(z)$ be a polynomial and let $R(z)$ be a rational function of the form*

$$R(z)=\sum_{j=p}^{q} a_j z^j \quad (a_p a_q \neq 0).$$

Assume that the composite function $P(R(z))$ is identically equal to the rational function $z^{-m}+z^n$, where m and n are two distinct natural numbers. Then the polynomial $P(z)$ is linear.

Lemma 1 is proved in [3]. Lemma 2 is elementary. However for completeness we shall show Lemma 2 at the end of this section.

We now prove the sufficiency. By the condition that the ratio A/B is not any real positive rational number, the two quantities A and B are clearly distinct. It then follows that

$$S(a^*z+b^*)=C_*(\exp(az)+\exp(bz)), \tag{3.1}$$

where $a^*=i(A-B)^{-1}$, $b^*=C(A-B)^{-1}$, $C_*=\exp(b^*A)$ and

$$a=Ai(A-B)^{-1}, \quad b=Bi(A-B)^{-1}. \tag{3.2}$$

Clearly $(a+b)(A-B)=i(A+B)$ and $a/b=A/B$. Hence by the condition on A and B, $a+b$ is different from zero and the ratio a/b is not any real positive rational number. For the sake of simplicity we set

$$F(z)=\exp(az)+\exp(bz).$$

It then follows from (3.2) that

$$F(z)=(1+\exp(-iz))\exp(az). \tag{3.3}$$

In particular $F(z)$ has only real simple zeros.

Suppose that this auxiliary entire function $F(z)$ admits a factorization

$$(3.4) \qquad F(z)=f(g(z))$$

with two entire functions $f(z)$ and $g(z)$. Our goal is to show that the left factor $f(z)$ or the right factor $g(z)$ is linear. We first consider the case where $f(z)$ is transcendental. For this case $f(z)$ has infinitely many zeros. Indeed this is trivial when $g(z)$ is a polynomial. If $g(z)$ is transcendental, by a result of Polya [6], the order of $f(z)$ must be zero. Hence $f(z)$ has infinitely many zeros. We denote the zeros of $f(z)$ with $\{w_n\}$. Then by the representation (3.4), all the roots of the equations $g(z)=w_n$ are surely zeros of $F(z)$, so that they are all real. It thus follows from Edrei's result [1] that the right factor $g(z)$ is a polynomial of degree at most two. Here we assume that $g(z)$ is quadratic. Then we may set $g(z)=z^2+sz$, where s is a constant. Evidently $g(z)=g(-z-s)$, so that

$$(3.5) \qquad (1+\exp(-iz))\exp(2az+as)=1+\exp(iz+is)$$

from (3.3) and (3.4). Let w be a zero of $f(z)$ and let z_1 and z_2 be the roots of the equation $g(z)=w$. Then $F(z_1)=F(z_2)=0$, so that $\exp(-iz_1)=\exp(-iz_2)=-1$ by (3.3) Therefore $\exp(is)=1$ because of $z_1+z_2=-s$. Hereby the above identity (3.5) becomes

$$\exp(2az+as)=\exp(iz).$$

It thus follows that $2a=i$, so that $a+b=0$ by (3.2). This is absurd by the condition. Consequently if the left factor $f(z)$ is transcendental, the right factor $g(z)$ must be linear.

We next consider the case where $f(z)$ is a polynomial of degree at least two. Since the auxiliary function $F(z)$ has simple zeros only, the left factor $f(z)$ has at least two zeros. We may assume without loss of generality that $f(0)=f(1)=0$. Then the right factor $g(z)$ has only real zeros and real ones. Hence by virtue of Lemma 1, either $g(z)$ is real entire or $g(z)$ is periodic with real period.

Let us consider the former case. In this case $g(z)$ has no finite asymptotic values. Since $f(z)$ is polynomial, the function $F(z)$ does not have any finite asymptotic values either. On the other hand it is clear from (3.3) that

$$|\exp(-ax)F(x)| \leqq 2$$

for all real values of x. Hereby the quantity a is purely imaginary. Hence from (3.3) again

$$F(iy) = |F(iy)| = (1+e^y)\exp(aiy) \tag{3.6}$$

for all real values of y. Thereby $F(iy)$ becomes infinite when y does positively infinite or negatively infinite. Here let us notice the reality of $g(z)$. Then $|g(iy)| = |g(-iy)|$, so that $g(iy)$ converges to infinity as y becomes infinite. It therefore follows that

$$\left|\frac{f(g(iy))}{f(g(-iy))}\right| \longrightarrow 1$$

when y tends to infinity. Accordingly from (3.4) and (3.6),

$$\exp(2aiy+y) \longrightarrow 1$$

as y becomes infinite. Consequently $2ai+1=0$, so that $a+b=0$ by (3.2). This is impossible.

It remains to consider the latter case where $g(z)$ is periodic with real period. Let u be a real period of $g(z)$. Then since $g(z)$ is of exponential type, by setting $c=2\pi i/u$, we can write $g(z)$ in the form

$$g(z) = \sum_{j=p}^{q} a_j(\exp(cz))^j, \tag{3.7}$$

where $a_p, a_{p+1}, \cdots, a_q$ are constants with $a_p a_q \neq 0$. On the other hand from (3.4), the auxiliary function $F(z)$ is also periodic with the same period u, that is, the identity $F(z+u)=F(z)$ holds. Hence $\exp(au)=\exp(bu)=1$, so that there are two nonzero integers m and n such that $a=-mc$ and $b=nc$. Of course, m and n are distinct because of $a+b\neq 0$. Accordingly $F(z)$ can be written in the form

$$F(z)=(\exp(cz))^{-m}+(\exp(cz))^{n}. \tag{3.8}$$

It thereby follows from (3.4), (3.7) and (3.8) that

$$w^{-m}+w^{n}=f(\sum_{j=p}^{q}a_j w^j)$$

for values of w. Thus by means of Lemma 2, the product mn must be nagative because $f(z)$ is a nonlinear polynomial. Hereby the ratio $a/b=-m/n$ is a real positive rational number. This contradicts the condition on a and b. Hence if the left factor $f(z)$ is polynomial, it must be linear. Consequently the auxiliary function $F(z)$ is surely prime in entire sense. Therefore by the relation (3.1), the entire function $S(z)$ is also prime in entire sense. This completes the proof.

Proof of Lemma 2. Let us set

$$P(z)=\sum_{j=0}^{k}b_j z^j \qquad (b_k\neq 0).$$

Hereafter we assume that the degree k is at least two, and obtain a contradiction. Since $P(R(z))=z^{-m}+z^{n}$ has a pole of order m at the origin, the integer p must be negative and $pk=-m$. Similarly since $P(R(z))$ has a pole of order n at the point at infinity, the integer q is positive and $qk=n$. Here we can suppose without loss of generality that $a_0=0$. It is clear that

$$\begin{aligned} z^{-m}+z^{n}&=b_k(a_p z^p)^k+b_k(a_q z^q)^k+\sum_{j=0}^{k-1}b_j(R(z))^j \\ &+b_k\sum_{j=1}^{k-1}\binom{k}{j}(a_p z^p)^{k-j}(R(z)-a_p z^p)^j \\ &+b_k\sum_{j=1}^{k}\binom{k}{j}(a_q z^q)^{k-j}(R(z)-a_p z^p-a_q z^q)^j. \end{aligned} \tag{3.9}$$

Assume that $R(z)-a_pz^p$ has a pole of order h at the origin. Of course by definition, $1 \leqq h < -p$. Then the fourth term of the right hand side of (3.9) has a pole of order $h+p-pk$ at the origin, while the fifth term has a pole of order kh at the origin. Clearly kh is less than $h+p-pk$. Hence the third term of the right hand side of (3.9) has a pole of order $h+p-pk$ at the origin. Hereby the order $h+p-pk$ is a multiple of p. This is absurd because of $1 \leqq h < -p$. Consequently $R(z)-a_pz^p$ is regular at the origin. Next let us assume that $R(z)-a_pz^p-a_qz^q$ has a pole of order l at the point at infinity. Then by the same reasoning as above, the third term of the right hand side of (3.9) must have a pole of order $l-q+qk$ at the infinity. Hereby the order $l-q+qk$ is a multiple of q. However this is clearly untenable by the fact $1 \leqq l < q$. Accordingly $R(z)-a_qz^q-a_pz^p$ is regular at the point at infinity. Therefore it is constant, so that $R(z)=a_pz^p+a_qz^q$ by the assumption $a_0=0$. Hence (3.9) becomes

$$b_k \sum_{j=1}^{k-1} \binom{k}{j} (a_p z^p)^{k-j} (a_q z^q)^j + \sum_{j=0}^{k-1} b_j (R(z))^j = 0.$$

Clearly the first term has a pole of order $q-p(k-1)$ at the origin, and at the point at infinity it has a pole of order $p+q(k-1)$. It thus follows that $q-p(k-1)$ is a multiple of p and $p+q(k-1)$ is a multiple of q. Hereby p/q and q/p are both integers. This yields that $-p=q$, so that $m=n$. We therefore have a contradiction. This completes the proof of Lemma 2.

References

[1] A. Edrei, Meromorphic functions with three radially distributed values, Trans. Amer. Math. Soc., 78 (1955), 276-293.
[2] F. Gross, Factorization of meromorphic functions, Math. Research Center, Washington D.C., 1972.
[3] T. Kobayashi, Some properties of real entire functions having only real zeros and real ones, Chiba Keiai University, 19 (1981), 513-543.
[4] T. Kobayashi, Factorization of certain entire functions, Chiba Keiai University, 26 (1984), 135-171.
[5] R. Nevanlinna, Eindeutige analytishe Funktionen, 2nd ed., Berlin, 1953.
[6] G. Polya, On an integral function of integral function, J. London Math. Soc., 1 (1926), 12-25.