

Factorization of certain entire functions

By Tadashi Kobayashi

The theory of factorization of meromorphic functions is a study of the ways in which a given meromorphic function can be represented as a composition of other meromorphic functions. Recently numerous results on the factorization theory have been obtained by many authors in several directions such as primeness, permutability and unique factorizability.

We say that two factorizations $f_1(g_1(z))$ and $f_2(g_2(z))$ of a meromorphic function, where $f_i(z)$ and $g_i(z)$ are meromorphic, are equivalent if and only if there exists a linear transformation $l(z)$ such that $f_2(z) = f_1(l(z))$ and $g_2(z) = l^{-1}(g_1(z))$, where $l^{-1}(z)$ stands for the inverse of $l(z)$. If every nontrivial factorization of a meromorphic function is equivalent to one another, then we say that this meromorphic function is uniquely factorizable. In the paper [4], Urabe proved the following result.

Let $f(z)$ be $z + h(e^z)$, and let $g(z)$ be $z + Q(e^z)$, where $h(z)$ is a nonconstant entire function and $Q(z)$ is a nonconstant polynomial. Assume that the order of the function $f(z)$ is finite. Then the composite function $f(g(z))$ is uniquely factorizable.

In order to prove this theorem, he used the next proposition as a key lemma.

PROPOSITION. *Let $F(z)$ be $z + H(z)$, where $H(z)$ is a nonconstant periodic entire function with period c . Suppose that $F(z)$ has a nontrivial factorization $f(g(z))$ with two entire functions $f(z)$ and $g(z)$. Then the right factor $g(z)$ has the form*

$$g(z) = az + h(z),$$

where a is a nonzero constant and $h(z)$ is a nonconstant periodic entire function with period c .

The purpose of this paper is to improve the above Urabe's result. We shall show the following theorem.

THEOREM. *Let $f(z)$ be a nonlinear entire function of finite lower order, and let $g(z)$ be a nonlinear entire function of exponential type. Suppose that the functions $f(z)-z$ and $g(z)-z$ are both periodic with the same period. Then the composite function $f(g(z))$ is uniquely factorizable.*

1. Proof of Proposition. First of all, for completeness, we give a proof of Proposition which is somewhat simpler than that in Urabe's paper, although the approach is similar.

Let $F(z)$ be $z+H(z)$, where $H(z)$ is a nonconstant periodic entire function of period c . Then it is clear that

$$(1.1) \quad F(z+c) - F(z) = c$$

for all values of z . Suppose that $F(z)$ is representable as $f(g(z))$ with nonlinear entire functions $f(z)$ and $g(z)$. It then follows from (1.1) that

$$f(g(z+c)) - f(g(z)) = c \neq 0$$

holds identically. In particular, the entire function $g(z+c) - g(z)$ has no zeros. Hence it is possible to take an entire function $s(z)$ such that

$$(1.2) \quad g(z+c) - g(z) = \exp(s(z))$$

for all values of z . Evidently we thus obtain the identity

$$(1.3) \quad g(z+2c) - g(z) = \exp(s(z)) + \exp(s(z+c)).$$

Furthermore from (1.1), $F(z+2c) - F(z) = 2c$ holds identically. Hereby the function $g(z+2c) - g(z)$ also has no zeros. It therefore follows from (1.3) that the entire function $\exp(s(z+c) - s(z))$ fails to take the values -1 and 0 . Accordingly this function reduces to a constant function. Hence with a suitable constant d ,

$$(1.4) \quad s(z+c) - s(z) = d$$

for all values of z . By virtue of these identities (1.2) and (1.4), for each natural number n , we have the relation

$$g(z+nc) - g(z) = \exp(s(z)) \sum_{j=0}^{n-1} \exp(jd).$$

If $e^d \neq 1$, then

$$g(z+nc) = g(z) + \exp(s(z)) \frac{1 - \exp(nd)}{1 - \exp(d)}$$

for all integers n and all values of z . Hereby the function $g(z)$ must be bounded on the half straight line $L^+ = \{x: x \geq 0\}$ or the half line $L^- = \{x: x \leq 0\}$. Since $F(z) = f(g(z))$, the function $F(z)$ must be also bounded on the line L^+ or the line L^- . However this is clearly absurd by the identity (1.1). Hence $e^d = 1$, so that

$$(1.5) \quad g(z+nc) = g(z) + n \exp(s(z))$$

for all values of z and all integers n .

Hereafter we want to show that the function $s(z)$ is constant. For this purpose, let us suppose that $s(z)$ is not constant. Then it is possible to find two points u and v such that

$$(1.6) \quad g(u) \neq g(v), \quad s(u) \neq s(v), \quad \exp(s(u)) = \exp(s(v)).$$

In fact, for the case $d \neq 0$, this is clear from (1.2), (1.4) and the fact $e^d = 1$. For the case $d = 0$, the function $s(z)$ is periodic of period c . Hence we can easily find two points u^* and v^* which satisfy $s(u^*) = s(u^* + c) = 2\pi i + s(v^*)$. If $g(u^*) = g(v^*)$, then $g(u^* + c) \neq g(v^*)$

by means of (1.2). Hereby there surely exist two points u and v satisfying the condition (1.6). Now let us consider the function $J(z)$ defined with

$$J(z) = \frac{\exp(s(z)) - \exp(s(v))}{g(z) - g(v)}.$$

By the condition $g(u) \neq g(v)$, it is possible to find a real positive number r such that $g(z) - g(v) \neq 0$ and $|s(z) - s(u)| \leq \pi$, in the closed disc $|z - u| \leq r$. Hence the function $J(z)$ is regular for the disc $|z - u| < r$ and $J(u) = 0$ from (1.6). Evidently $J(z)$ does not vanish identically, since $s(z)$ is not constant by assumption. Thereby for all sufficiently large natural numbers m , there exist points z_m satisfying $|z_m - u| < r$ and $J(z_m) = -1/m$. We thus have

$$(1.7) \quad g(z_m) + m \exp(s(z_m)) = g(v) + m \exp(s(v))$$

for these points z_m . Let E be the compact set

$$\{tz + (1-t)v : 0 \leq t \leq 1, |z - u| \leq r\}.$$

Then we can set the quantities

$$K = \max |g(z)|, \quad L = \min |\exp(s(z))|,$$

where z ranges over the compact set E . Here we choose a natural number m^* so large that $m^* \geq 2K/L$ and for each integer m with $m \geq m^*$, there exists a point z_m satisfying $|z_m - u| \leq r$ and the above condition (1.7).

Now for each number m with $m \geq m^*$, we can define the curve C_m given by

$$(1.8) \quad \begin{aligned} z_m(t) &= g(l_m(t)) + m \exp(s(l_m(t))), \\ l_m(t) &= tz_m + (1-t)v \end{aligned}$$

for $0 \leq t \leq 1$. Clearly by virtue of (1.7), $z_m(0) = z_m(1)$, so that the curve C_m is closed. Furthermore by definition, the segment $l_m(t)$

between the points v and z_m is contained in the set E . Hence we have

$$(1.9) \quad |z_m(t)| \geq mL - K > 0$$

for $0 \leq t \leq 1$. Thereby this curve C_m does not pass through the origin. Besides, we also obtain

$$\left| \frac{z_m(t)}{m \exp(s(l_m(t)))} - 1 \right| = \left| \frac{g(l_m(t))}{m \exp(s(l_m(t)))} \right| \leq \frac{K}{mL}$$

for $0 \leq t \leq 1$. Since $m \geq m^* \geq 2K/L$, we can define the argument of the curve C_m so that

$$(1.10) \quad \left| \arg z_m(t) - \operatorname{Im}(s(l_m(t))) \right| \leq \frac{\pi}{3}$$

for all real values of t with $0 \leq t \leq 1$. Since $|\operatorname{Im}(s(u) - s(v))| \geq 2\pi$ from (1.6), $|\operatorname{Im}(s(z_m) - s(v))| \geq \pi$. It thus follows from (1.10) that

$$\arg z_m(1) - \arg z_m(0) \neq 0.$$

Hence the winding number of the curve C_m with respect to the origin is different from zero. On the other hand from the identity (1.1),

$$F(z + mc) - F(z) = mc$$

for all values of z and all integers m . It hence follows from (1.2) and (1.5) that

$$f(g(z) + m \exp(s(z))) = F(z) + mc,$$

so that from the definition (1.8),

$$f(z_m(t)) = F(l_m(t)) + mc$$

for real values of t with $0 \leq t \leq 1$ and every integer m with $m \geq m^*$. Hence by means of (1.9), the maximum modulus principle yields

$$M(mL-K, f) \leq K^* + |c|m$$

for $m \geq m^*$, where $M(r, f)$ stands for the maximum modulus function of $f(z)$ and K^* is the maximum of $|F(z)|$ on the compact set E . This means that the left factor $f(z)$ must be linear. We thus arrive at a contradiction. Therefore the function $s(z)$ must be constant, so that the right factor $g(z)$ satisfies the identity

$$g(z+c) - g(z) = c^*$$

with a suitable nonzero constant c^* . This completes the proof of Proposition.

2. Preliminary results. Let $f(z)$ be a regular function in the punctured unit disc $0 < |z| < 1$. Suppose that $f(z)$ has a pole of order n at the origin. Then $z^n f(z)$ converges to some nonzero value c as z approaches the origin. Here we may assume without loss of generality that this value c is real positive. Then we can take a real number r^* with $0 < r^* < 1$ such that

$$(2.1) \quad \left| z^n f(z) - c \right| < \frac{c}{2}$$

for all values of z with $0 < |z| \leq r^*$. Hence we have

$$(2.2) \quad |z|^n \operatorname{Re}(f(z)) < -\frac{c}{2}$$

for all values of z which satisfy $0 < |z| \leq r^*$ and $\arg z = (2j-1)\pi/n$ ($j=1, \dots, n$). Now let us consider the auxiliary function given by

$$f_*(z) = \frac{1}{z} \exp(f(z))$$

for $0 < |z| < 1$. Then this function $f_*(z)$ is regular and never vanishes for $0 < |z| < 1$. Furthermore $f_*(z)$ satisfies

$$\log |f_*(z)| = -\log |z| + \operatorname{Re}(f(z))$$

there. It thus follows from (2.2) that

$$2r^n \log |f_*(z)| + c + 2r^n \log r \leq 0$$

for all values of z satisfying $0 < r = |z| \leq r^*$ and $\arg z = (2j-1)\pi/n$ ($j=1, \dots, n$). This means that the function $f_*(z)$ converges to zero as z approaches the origin along the half rays $\arg z = (2j-1)\pi/n$ ($j=1, \dots, n$). For each integer j with $1 \leq j \leq n$, we define the sector

$$A_j = \{z: 0 < |z| < r^*, |\arg z - 2j\pi/n| < \pi/n\}.$$

Then the function $f_*(z)$ is bounded on the boundaries of these sectors A_j except for the origin. From (2.1), it also follows that

$$2 \log |f_*(z)| + 2 \log r \geq cr^{-n}$$

for values of z satisfying $0 < r = |z| \leq r^*$ and $\arg z = 2j\pi/n$ ($j=1, \dots, n$), so that the function $f_*(z)$ becomes infinite as z approaches the origin along the radial lines $\arg z = 2j\pi/n$. Therefore the classical Lindelöf's theorem yields that $f_*(z)$ takes every finite value other than zero in each sector A_j .

Here we take a real positive number R such that

$$(2.3) \quad |f_*(z)| < R$$

for each point of the boundaries of the sectors A_j except for the origin and $f'_*(z) \neq 0$ at every point of these sectors A_j which satisfies $|f_*(z)| = R$. Then in each sector A_j , we can take a point z_j at which $f_*(z_j) = R$. Of course, $f'_*(z_j) \neq 0$. Hence we can define the curve $C_j: z = z_j(t)$ ($a_j < t < b_j$, $-\infty \leq a_j < 0 < b_j \leq +\infty$) satisfying $z_j(0) = z_j$ and

$$(2.4) \quad f_*(z_j(t)) = Re^{it}$$

for $a_j < t < b_j$. By virtue of (2.3), the curve C_j is contained entirely in the sector A_j , and if this curve C_j intersects itself, then $f_*(z)$ reduces

to a constant function. Thereby C_j is a simple analytic curve lying in the sector A_j . For a moment suppose that the curve C_j intersects some circle $|z|=r$ ($0 < r < r^*$) infinitely many times. Then the function

$$F(z) = \overline{f_*(r^2/\bar{z})} f_*(z)$$

which is regular for $r^2 < |z| < 1$, has infinitely many points of the circle $|z|=r$ satisfying the equation $F(z)=R^2$. This implies that $F(z)=R^2$ holds identically, so that $|f_*(z)|=R$ for $|z|=r$. This is absurd by (2.3). Thereby the curve C_j intersects every circle of center at the origin at most finitely many times. By these facts, we can suppose that C_j reaches to the origin when the real parameter t converges to a_j . Hence we find

$$(z_j(t))^n f(z_j(t)) \rightarrow c$$

as t tends to a_j . Since the limit c is real positive, we therefore have

$$(2.5) \quad \operatorname{Re}((z_j(t))^n) \operatorname{Re}(f(z_j(t))) - \operatorname{Im}((z_j(t))^n) \operatorname{Im}(f(z_j(t))) \rightarrow c,$$

$$(2.6) \quad \operatorname{Re}((z_j(t))^n) \operatorname{Im}(f(z_j(t))) + \operatorname{Im}((z_j(t))^n) \operatorname{Re}(f(z_j(t))) \rightarrow 0$$

as t converges to a_j . On the other hand it is clear from (2.4) that

$$(2.7) \quad \begin{aligned} \log R = \log |f_*(z_j(t))| &= -\log |z_j(t)| + \operatorname{Re}(f(z_j(t))), \\ \operatorname{Im}(f(z_j(t))) - \arg z_j(t) &= t + 2\pi m_j \end{aligned}$$

for all real values of t with $a_j < t < b_j$, where m_j is a suitable integer and

$$(2.8) \quad |\arg z_j(t) - 2j\pi/n| < \pi/n$$

for $a_j < t < b_j$. It therefore follows from (2.7) that

$$|z_j(t)|^n \operatorname{Re}(f(z_j(t))) \rightarrow 0$$

as t tends to a_j , so that the above (2.5) and (2.6) become

$$(2.9) \quad \operatorname{Im}((z_j(t))^n) \operatorname{Im}(f(z_j(t))) \rightarrow -c,$$

$$(2.10) \quad \operatorname{Re}((z_j(t))^n) \operatorname{Im}(f(z_j(t))) \rightarrow 0$$

as t tends to a_j , respectively. In particular, $\operatorname{Im}(f(z_j(t)))$ becomes infinite as t converges to a_j . Hence with the help of (2.7) and (2.8), the quantity a_j must be negatively infinite. Furthermore from (2.7), it is possible to take a real number t^* such that

$$\operatorname{Im}(f(z_j(t))) < 0$$

for all values of t with $-\infty < t \leq t^*$. Hereby the relation (2.9) yields

$$\operatorname{Im}((z_j(t))^n) > 0$$

for $-\infty < t \leq t_*(\leq t^*)$, since the limit c is real positive by assumption. On taking account of (2.8), we thus obtain

$$(2.11) \quad 2j\pi/n \leq \arg z_j(t) \leq (2j+1)\pi/n$$

for $-\infty < t \leq t_*$. In addition to these facts, it is clear from (2.9) and (2.10) that

$$\frac{\operatorname{Re}((z_j(t))^n)}{\operatorname{Im}((z_j(t))^n)} \rightarrow 0$$

as t tends to $-\infty$. It therefore follows from (2.11) that

$$\arg z_j(t) \rightarrow (4j+1)\pi/2n$$

as t converges to $-\infty$. Quite similarly, we can conclude that the quantity b_j is positively infinite and the argument of $z_j(t)$ satisfies

$$\arg z_j(t) \rightarrow (4j-1)\pi/2n$$

as the real parameter t grows to infinity.

Accordingly we have proved the following assertion.

LEMMA 1. Let $f(z)$ be a regular function in the punctured unit disc $0 < |z| < 1$. Suppose that $f(z)$ has the series expansion

$$f(z) = \sum_{j \geq -n} a_j z^j,$$

where $n \geq 1$ and $a_{-n} \neq 0$. Then for each natural number j with $1 \leq j \leq 2n$, there exists a simple curve $C_j: z = z_j(t)$ ($0 \leq t < +\infty$) in this annulus satisfying

$$\lim_{t \rightarrow +\infty} z_j(t) = 0,$$

$$\lim_{t \rightarrow +\infty} \arg z_j(t) = \frac{2j-1}{2n} \pi + \frac{a^*}{n},$$

$$|\exp(f(z_j(t)))| = R |z_j(t)|$$

for all values of t , where R is a real positive constant and $a^* = \arg a_{-n}$ with $0 \leq a^* < 2\pi$.

Let $P(z)$ and $Q(z)$ be regular functions in $0 < |z| < +\infty$ which satisfy

$$(2.12) \quad z^{-m} P(z) \rightarrow a, \quad z^{-n} Q(z) \rightarrow b$$

as z tends to infinity, where a, b are both nonzero constants and m, n are integers. Here we suppose that $m \leq n$ and $n \geq 1$. Then by virtue of Lemma 1, for each integer j with $1 \leq j \leq 2n$, we can take a simple curve $C_j^*: z = z_j(t)$ ($0 \leq t < +\infty$) in the annulus $0 < |z| < +\infty$ which satisfies

$$(2.13) \quad \lim_{t \rightarrow +\infty} z_j(t) = \infty,$$

$$(2.14) \quad \lim_{t \rightarrow +\infty} \arg z_j(t) = \frac{2j-1}{2n} \pi - \frac{1}{n} \arg b,$$

where $0 \leq \arg b < 2\pi$, and with a real positive constant R

$$(2.15) \quad |z_j(t) \exp(Q(z_j(t)))| = R$$

for all real values of t with $t \geq 0$.

Let us set

$$(2.16) \quad w_j(t) = z_j(t) \exp(P(z_j(t)))$$

for $0 \leq t < +\infty$ ($j=1, 2, \dots, 2n$), and let us consider the asymptotic behaviors of these functions. By the definition (2.16), it is clear that

$$(2.17) \quad \log |w_j(t)| - \log |z_j(t)| = \operatorname{Re}(P(z_j(t)))$$

for $0 \leq t < +\infty$, $1 \leq j \leq 2n$. On the other hand with the help of (2.12), (2.13) and (2.14), it is also clear that

$$(2.18) \quad \frac{\operatorname{Re}(P(z_j(t)))}{|z_j(t)|^m} \longrightarrow |a| \cos(s^* + jm\pi/n)$$

as t grows to infinity, where s^* is the real number given by

$$(2.19) \quad s^* = \arg a - \frac{m}{n} \arg b - \frac{m}{2n} \pi$$

with $0 \leq \arg a < 2\pi$, $0 \leq \arg b < 2\pi$.

Assume first that the integer m is negative. Then by means of (2.13), the denominator of (2.18) converges to zero as t tends to infinity. Hence the real part of $P(z_j(t))$ also converges to zero as t becomes infinite. It thus follows from (2.17) that

$$(2.20) \quad |w_j(t)| \sim |z_j(t)|$$

as t grows to infinity ($j=1, 2, \dots, 2n$). Assume next that the number m is equal to zero. Then it is clear that $P(z_j(t))$ converges to the value a when t tends to infinity. Hence from (2.17) again, we at once have

$$(2.21) \quad |w_j(t)| \sim |e^a| |z_j(t)|$$

as t becomes infinite ($j=1, 2, \dots, 2n$).

For the case where $1 \leq m < n$, it is possible to find a number j among the natural numbers from 1 to $2n$ which satisfies

$$(2.22) \quad \cos(s^* + jm\pi/n) > 0.$$

In fact, since $\exp(im\pi/n) \neq 1$,

$$\sum_{k=1}^{2n} (\exp(im\pi/n))^k = 0.$$

Hence we obtain

$$\sum_{k=1}^{2n} \cos(s^* + km\pi/n) = 0.$$

Thereby because of $m/n < 1$, $\cos(s^* + km\pi/n) \neq 0$ for some integer k with $1 \leq k \leq 2n$. Hence there surely exists such a number j . Let j be an integer satisfying (2.22). Then by means of (2.17) and (2.18),

$$\frac{\log |w_j(t)|}{|z_j(t)|^m} \rightarrow |a| \cos(s^* + jm\pi/n) > 0$$

as t tends to infinity. It thus follows that

$$(2.23) \quad \frac{|z_j(t)|^n}{|w_j(t)|} \rightarrow 0$$

as t grows to infinity.

It remains to treat the case $m=n$. For this case it is clear that

$$\cos(s^* + jm\pi/n) = (-1)^j \cos s^*$$

for all integers j . Hence if $\cos s^* \neq 0$, then there surely exists an integer j which satisfies (2.22). Hereby the above asymptotic relation (2.23) holds for this integer j . Suppose that $\cos s^* = 0$. Then from the definition (2.19), the ratio a/b must be real. We denote this real ratio a/b by r . Then by virtue of (2.15), the relation (2.17) becomes

$$\begin{aligned}
 \log |w_j(t)| &= \log |z_j(t)| + r \operatorname{Re}(Q(z_j(t))) \\
 &\quad + \operatorname{Re}(P(z_j(t)) - rQ(z_j(t))) \\
 (2.24) \qquad &= (1-r) \log |z_j(t)| + r \log R \\
 &\quad + \operatorname{Re}(P(z_j(t)) - rQ(z_j(t)))
 \end{aligned}$$

for $0 \leq t < +\infty$, $1 \leq j \leq 2n$. Of course, the function $P(z) - rQ(z)$ is regular or has a pole of order at most $n-1$ at the point at infinity. Suppose that $P(z) - rQ(z)$ has a pole of order k at the infinity. Then the order k is less than n . Hence we are able to take an integer j among the integers from 1 to $2n$ such that the quantity

$$\frac{\operatorname{Re}(P(z_j(t)) - rQ(z_j(t)))}{|z_j(t)|^k}$$

converges to some real positive value c^* as t becomes infinite. It therefore follows from (2.24) that for this number j ,

$$\frac{\log |w_j(t)|}{|z_j(t)|^k} \rightarrow c^*$$

as t grows to infinity. Hereby the asymptotic relation (2.23) also holds in this case. If the function $P(z) - rQ(z)$ is regular at the infinity, then the above (2.24) yields

$$(2.25) \qquad \frac{\log |w_j(t)|}{\log |z_j(t)|} \rightarrow 1-r$$

as t converges to infinity ($j=1, 2, \dots, 2n$). Hence if $r < 1$, then $w_j(t)$ approaches the infinity as t grows to infinity. If $r > 1$, then $w_j(t)$ converges to zero as t becomes infinite.

We are now in a position to prove the next result.

LEMMA 2. *Let $P(z)$, $Q(z)$, $A(z)$ and $B(z)$ be regular functions in the annulus $0 < |z| < +\infty$. Suppose that the functions $P(z)$ and $Q(z)$ satisfy*

$$z^{-m}P(z) \rightarrow a, \quad z^{-n}Q(z) \rightarrow b$$

as z tends to infinity, where a, b are both nonzero constants and m, n are integers with $m \leq n$ and $n \geq 1$. Suppose further that the functional equation

$$A(ze^{P(z)}) - B(ze^{Q(z)}) = P(z) - Q(z)$$

holds identically in this annulus. Then if $P(z) - Q(z)$ is not regular at the infinity, for some natural number k , either the function $z^k A(z)$ converges to zero as z approaches the origin along some asymptotic curve tending to the origin or the function $z^{-k} A(z)$ converges to zero as z approaches the point at infinity along some asymptotic curve tending to the infinity.

Proof. For each natural number j from 1 to $2n$, we have defined the simple asymptotic curve $C_j^*: z = z_j(t)$ ($0 \leq t < +\infty$) satisfying the conditions (2.13), (2.14) and (2.15). Inserting the functions $z_j(t)$ into the functional equation and using the notation (2.16), we deduce

$$\begin{aligned} A(w_j(t)) - B(z_j(t) \exp(Q(z_j(t)))) \\ = P(z_j(t)) - Q(z_j(t)) \end{aligned}$$

for $0 \leq t < +\infty$, $1 \leq j \leq 2n$. Hence by virtue of the condition (2.15), the quantities

$$A(w_j(t)) - P(z_j(t)) + Q(z_j(t))$$

must be bounded for $0 \leq t < +\infty$. It thus follows from (2.12) and (2.13) that

$$(2.26) \quad \frac{A(w_j(t))}{(z_j(t))^n} \rightarrow c$$

as t grows to infinity, where the limit c is equal to $-b$ in the case $m < n$, and c is $a - b$ for the case $m = n$ ($j = 1, 2, \dots, 2n$).

Suppose first that $m \leq 0$. Then for every number j , the asymptotic behavior (2.20) or (2.21) holds. Hence from (2.26), it is clear that

$$\frac{A(w_j(t))}{(w_j(t))^{n+1}} \rightarrow 0$$

as t converges to infinity. This means that the function $z^{-n-1}A(z)$ converges to zero as z approaches the point at infinity along the asymptotic curves $z=w_j(t)$ ($0 \leq t < +\infty$). Next let us consider the case $1 \leq m < n$. In this case the asymptotic relation (2.23) holds for some number j . Hence for this number j ,

$$(2.27) \quad \frac{A(w_j(t))}{w_j(t)} \rightarrow 0$$

as t grows to infinity, so that as z approaches the point at infinity along the asymptotic curve $z=w_j(t)$ ($0 \leq t < +\infty$), the function $z^{-1}A(z)$ converges to zero. Finally we treat the case $m=n$. Assume that the ratio a/b is not real. Then by what is mentioned above, the relation (2.23) is also true. Thereby the above (2.27) also holds. Accordingly we have the desired result in this case. Assume next that the ratio $r=a/b$ is real. If $P(z)-rQ(z)$ has a pole at the infinity, then the relation (2.23) is still valid for some number j . Hence the desired result follows immediately. If $P(z)-rQ(z)$ is regular at the infinity, then the asymptotic behavior (2.25) holds for every number j . Suppose that $r < 1$. Then from (2.25),

$$\frac{(z_j(t))^n}{(w_j(t))^k} \rightarrow 0$$

as t grows to infinity for all integers k satisfying $k(1-r) > n$. Hence (2.26) yields that

$$\frac{A(w_j(t))}{(w_j(t))^k} \rightarrow 0$$

as t tends to infinity ($j=1, 2, \dots, 2n$). Therefore the function $z^{-k}A(z)$ surely has the value zero as an asymptotic value at the point at infinity.

Suppose that $r > 1$. Then by means of (2.25),

$$(w_j(t))^k (z_j(t))^n \longrightarrow 0$$

as t grows to infinity for all integers k satisfying $n + (1-r)k < 0$. Hereby from (2.26) again,

$$(w_j(t))^k A(w_j(t)) \longrightarrow 0$$

when t becomes infinite ($j=1, 2, \dots, 2n$). This means that the function $z^k A(z)$ converges to zero as z approaches the origin along the asymptotic curves $z = w_j(t)$ ($0 \leq t < +\infty$). This completes the proof.

We conclude the section by proving the following fact.

LEMMA 3. *Let $E(z)$ be a regular function in the punctured plane $0 < |z| < +\infty$ which has an essential singularity at the infinity. Suppose that $E(z)$ converges to zero as z approaches the infinity along some asymptotic curve. Then the function $H(z) = E(e^z)$ satisfies*

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \log T(r, H) > 0,$$

where $T(r, H)$ denotes the characteristic function of $H(z)$.

Proof. By Cauchy's integral theorem, the function $E(z)$ can be written as a sum $E^*(z) + E_*(1/z)$, where $E^*(z)$ and $E_*(z)$ are both entire functions with $E_*(0) = 0$. Of course, by assumption, $E^*(z)$ does not reduce to a polynomial. Furthermore by assumption, this transcendental entire function $E^*(z)$ has a finite asymptotic value. Hence by virtue of Edrei's theorem [1] which generalizes Wiman's classical theorem, the lower order of $E^*(z)$ is at least $1/2$. We can thus choose a positive number ε such that

$$(2.28) \quad \log T(r, E^*(z)) \geq \varepsilon \log r$$

for all sufficiently large values of r . On the other hand it is clear that

$$|H(z)| \geq |E^*(e^z)| - |E_*(e^{-z})|$$

for all values of z . Since $E^*(e^z)$ is bounded in the left half plane $z=x+iy, x<0$, we at once obtain

$$(2.29) \quad M(r, E^*(e^z)) \leq M(r, H(z)) + O(1)$$

for values of r . Here let us estimate $M(r, E^*(e^z))$ from below. By the maximum modulus principle [2], it follows that

$$(2.30) \quad \begin{aligned} M(r, E^*(e^z)) &\geq M(R(r), E^*(z)), \\ R(r) &= AM(r/2, e^z) - A - 1 \end{aligned}$$

for sufficiently large values of r , where A is a positive absolute constant. Since $M(r/2, e^z) = \exp(r/2)$, we therefore obtain from (2.29) and (2.30) that

$$(2.31) \quad M(\exp(r/3), E^*(z)) \leq M(r, H(z)) + O(1)$$

for all sufficiently large values of r . On combining this (2.31) with (2.28), we obtain the desired result immediately.

3. Further results on the functional equation. Let $P(z)$ and $Q(z)$ be nonconstant regular functions in the annulus $0 < |z| < +\infty$ which satisfy

$$(3.1) \quad z^{-m}P(z) \longrightarrow a, \quad z^{-n}Q(z) \longrightarrow b$$

as z tends to infinity, where a, b are nonzero constants and m, n are integers with $m \leq n$ and $n \geq 1$. Let $A(z)$ and $B(z)$ be also nonconstant regular functions in the annulus $0 < |z| < +\infty$. Hereafter we suppose that the functional equation

$$(3.2) \quad P(z) - Q(z) = A(ze^{P(z)}) - B(ze^{Q(z)})$$

holds identically in this annulus. In the previous section we have considered and obtained some results on this functional equation. However we need further consideration to prove our theorem. Let us begin by showing the following fact.

LEMMA 4. *If $P(z) - Q(z)$ is regular at the point at infinity, then $P(z) - Q(z)$ reduces to a constant function.*

Proof. Since $P(z)$ has a pole at the point at infinity, the function $z \exp(P(z))$ has an essential singularity at the infinity. Hence for an arbitrary value w other than zero, there exists a sequence $\{z_n\}$ tending to infinity such that $z_n \exp(P(z_n)) = w$ ($n=1, 2, \dots$). It thus follows from the functional equation (3.2) that

$$(3.3) \quad P(z_n) - Q(z_n) = A(w) - B(z_n \exp(Q(z_n)))$$

for $n \geq 1$. Since $P(z) - Q(z)$ has a finite limit c as z tends to infinity, $P(z_n) - Q(z_n)$ and $z_n \exp(Q(z_n))$ respectively converge to c and $w \exp(-c)$ as n grows to infinity. Therefore from (3.3),

$$A(w) - B(c^*w) = c$$

with $c^* = \exp(-c)$. Since w is arbitrary, the functions $A(z)$ and $B(z)$ satisfy the identity

$$(3.4) \quad A(z) = B(c^*z) + c.$$

Let s be a point in the annulus $0 < |z| < +\infty$ satisfying

$$\exp(Q(s) - P(s)) = c^*.$$

Then it is clear that $c^*s \exp(P(s)) = s \exp(Q(s))$, so that from (3.4),

$$\begin{aligned} A(s \exp(P(s))) &= c + B(c^*s \exp(P(s))) \\ &= c + B(s \exp(Q(s))). \end{aligned}$$

We thus obtain from (3.2) that $P(s) - Q(s) = c$. This means that all

the roots of the equation $\exp(Q(z) - P(z)) = c^*$ are also roots of the equation $P(z) - Q(z) = c$. It is a contradiction unless $P(z) - Q(z)$ is constant. Lemma 4 is now proved.

Now let us consider the functional equation (3.2) under the additional condition that $A(z)$ is regular or has a pole at the point at infinity.

If the integer m is negative or zero, then $P(z)$ is regular at the infinity by (3.1). Hence $A(z \exp(P(z)))$ is also regular or has a pole there. Hereby from (3.2), the function $B(z \exp(Q(z)))$ has a finite or infinite limit as z tends to infinity. However this is absurd because $z \exp(Q(z))$ has an essential singularity at the point at infinity.

Next we consider the case $1 \leq m < n$. By what is shown in the previous section, we can define a curve $C: z = z(t)$ ($0 \leq t < +\infty$) tending to the infinity such that with a positive constant R ,

$$(3.5) \quad |z(t) \exp(Q(z(t)))| = R$$

for all values of t and

$$(3.6) \quad \frac{|z(t)|^n}{|w(t)|} \longrightarrow 0$$

as t becomes infinite, where $w(t) = z(t) \exp(P(z(t)))$. Inserting this function $z(t)$ into (3.2) and taking account of (3.1), (3.5) and (3.6), we at once have that $A(w(t))/w(t)$ converges to zero when t grows to infinity. This implies that the function $A(z)$ must be regular at the point at infinity. Consequently the right hand side of the functional equation (3.2) must be bounded on the curve C . However this is clearly absurd by the assumption $m < n$.

From now on we shall treat the case $m = n$. For the sake of simplicity we set $0 \leq a^* = \arg a < 2\pi$, $0 \leq b^* = \arg b < 2\pi$, $a_* = |a|$ and $b_* = |b|$. Firstly we consider the case $\cos(a^* - b^*) \neq -1$. Since

$$\begin{aligned} 2\cos(x + a^*)\cos(x + b^*) &= \cos(2x + a^* + b^*) + \cos(a^* - b^*) \\ &> \cos(2x + a^* + b^*) - 1 \end{aligned}$$

for all real values of x , it is possible to find a real number c^* such that

$$\cos(nc^* + a^*) > 0, \quad 0 < \cos(nc^* + b^*) < 1.$$

Because of continuity, we can also find a small positive number t^* such that

$$(3.7) \quad \cos(nt + a^*) > 0, \quad 0 < \cos(nt + b^*) < 1$$

for all real values of t with $|t - c^*| \leq t^*$. Of course, this number t^* satisfies $0 < 2nt^* < \pi/2$. Here we choose a positive number ε so small that

$$(3.8) \quad a_* \cos(nt + a^*) > \varepsilon,$$

$$(3.9) \quad b_* \cos(nt + b^*) > 2\varepsilon,$$

$$(3.10) \quad b_* |\sin(nt + b^*)| > \varepsilon$$

for all real values of t with $|t - c^*| \leq t^*$, and

$$(3.11) \quad b_* |\cos(nc^* + b^*) - \cos(nc^* \pm nt^* + b^*)| > 2\varepsilon,$$

$$(3.12) \quad b_* |\sin(nc^* + b^*) - \sin(nc^* \pm nt^* + b^*)| > 3\varepsilon.$$

Since $0 < 2nt^* < \pi/2$ and $\sin(nt + b^*) \neq 0$ for $|t - c^*| \leq t^*$ by (3.7), we can surely choose such a positive number ε . By referring to (3.1), for this number ε , we can also choose a positive number $R^* > 1$ such that

$$|P(z) - az^n| \leq \varepsilon |z|^n, \quad |Q(z) - bz^n| \leq \varepsilon |z|^n$$

for all values of z with $R^* \leq |z| < +\infty$. It thus follows that

$$(3.13) \quad \begin{aligned} & |\operatorname{Re} P(re^{it}) - a_* r^n \cos(nt + a^*)| \leq \varepsilon r^n, \\ & |\operatorname{Re} Q(re^{it}) - b_* r^n \cos(nt + b^*)| \leq \varepsilon r^n, \\ & |\operatorname{Im} Q(re^{it}) - b_* r^n \sin(nt + b^*)| \leq \varepsilon r^n \end{aligned}$$

for all real values of t and r with $r \geq R^*$. Evidently by virtue of (3.9)

and (3.13),

$$(3.14) \quad \log |z \exp(Q(z))| \geq \log r + \varepsilon r^n$$

for all values of $z = re^{it}$ with $r \geq R^*$ and $|t - c^*| \leq t^*$. In particular the function $z \exp(Q(z))$ converges uniformly to infinity as z tends to infinity from the inside of the sector $|\arg z - c^*| \leq t^*$. Furthermore from (3.8), (3.9) and (3.13), we can take a natural number k such that $0 \leq \operatorname{Re} P(z) \leq k \operatorname{Re} Q(z)$, so that

$$(3.15) \quad |z| \leq |z \exp(P(z))| \leq |z \exp(Q(z))|^k$$

in the sector $|z| \geq R^*$, $|\arg z - c^*| \leq t^*$.

Let s and u be points which satisfy $s_* = |s| > R^*$, $u_* = |u| > R^*$, $\arg s = c^*$, $\arg u = c^* - t^*$ and $|s \exp(Q(s))| = |u \exp(Q(u))|$. Then by means of (3.13),

$$\begin{aligned} & (b_* \cos(nc^* - nt^* + b^*) - \varepsilon) u_*^n - (b_* \cos(nc^* + b^*) + \varepsilon) s_*^n \\ (3.16) \quad & \leq \operatorname{Re}(Q(u) - Q(s)) = \log s_* - \log u_* \\ & \leq (b_* \cos(nc^* - nt^* + b^*) + \varepsilon) u_*^n - (b_* \cos(nc^* + b^*) - \varepsilon) s_*^n, \\ & (b_* \sin(nc^* - nt^* + b^*) - \varepsilon) u_*^n - (b_* \sin(nc^* + b^*) + \varepsilon) s_*^n \\ (3.17) \quad & \leq \operatorname{Im}(Q(u) - Q(s)) \\ & \leq (b_* \sin(nc^* - nt^* + b^*) + \varepsilon) u_*^n - (b_* \sin(nc^* + b^*) - \varepsilon) s_*^n. \end{aligned}$$

Assume for a moment that $\sin(nc^* + b^*) > 0$. Then it is clear from (3.7), (3.11) and (3.12) that

$$(3.18) \quad \cos(nc^* - nt^* + b^*) - \cos(nc^* + b^*) > 2\varepsilon/b_*,$$

$$(3.19) \quad \sin(nc^* + b^*) - \sin(nc^* - nt^* + b^*) > 3\varepsilon/b_*.$$

Hence if $u_* \geq s_*$, then

$$(b_* \cos(nc^* - nt^* + b^*) - b_* \cos(nc^* + b^*) - 2\varepsilon) s_*^n \leq 0$$

from (3.9) and (3.16). This is absurd by (3.18). Hereby $u_* \leq s_*$, so that (3.17) and (3.19) yield

$$(3.20) \quad \operatorname{Im} (Q(u) - Q(s)) < -\varepsilon s_*^n.$$

On the contrary assume that $\sin(nc^* + b^*) < 0$. Then from (3.7), (3.11) and (3.12) again,

$$(3.21) \quad \cos(nc^* + b^*) - \cos(nc^* - nt^* + b^*) > 2\varepsilon/b_*$$

and (3.19) holds good. Hence by the same fashion as above, we have $u_* \geq s_*$ from (3.16) and (3.21). Since

$$b_* \sin(nc^* - nt^* + b^*) + \varepsilon < 0$$

by (3.10), we therefore obtain (3.20). Consequently the inequality (3.20) is always true. Quite similarly for a point v which satisfies $|v| > R^*$, $\arg v = c^* + t^*$ and $|s \exp(Q(s))| = |v \exp(Q(v))|$, we can see that

$$(3.22) \quad \operatorname{Im} (Q(v) - Q(s)) > \varepsilon s_*^n.$$

Now let R be an arbitrary positive number with $R > R^*$ and $\varepsilon R^n > 4\pi$, and let z^* be a point of the ray $\arg z = c^*$ such that $|z^*| > R$ and

$$(3.23) \quad R_* = |z^* \exp(Q(z^*))| > \max |z \exp(Q(z))|,$$

where the maximum is taken over the circle $|z| = R$. By referring to (3.14), such a point z^* surely exists and $R_* > R$. With this point z^* , we consider a curve $C^*: z = z(t)$ ($0 \leq t \leq 2\pi$) satisfying $z(0) = z^*$ and

$$(3.24) \quad z(t) \exp(Q(z(t))) = z^* \exp(Q(z^*)) e^{it}$$

for $0 \leq t \leq 2\pi$. By the condition (3.23), this curve C^* lies entirely in $|z| > R$. Furthermore the argument of $z(t)$ satisfies

$$\arg z(t) = c^* + t + \operatorname{Im}(Q(z^*) - Q(z(t)))$$

for $0 \leq t \leq 2\pi$. Hence if C^* intersects the ray $\arg z = c^* - t^*$ at some point $u = z(t')$, then $c^* - t^* = c^* + t' + \operatorname{Im}(Q(z^*) - Q(u))$, so that

$$|\operatorname{Im}(Q(z^*) - Q(u))| \leq t^* + 2\pi < 3\pi.$$

However since $\varepsilon|z^*|^n > 4\pi$, this is absurd by (3.20). Similarly by means of (3.22), the curve C^* never intersects the half line $\arg z = c^* + t^*$. By these facts and by the property (3.14), we can conclude that the curve C^* satisfying (3.24) is surely well defined and it is contained entirely in the sector $|\arg z - c^*| \leq t^*$. Since $|\arg z(t) - c^*| \leq t^*$ and $|z(t)| > R > R^*$ for $0 \leq t \leq 2\pi$, it follows from (3.15) and (3.24) that

$$(3.25) \quad \begin{aligned} R^* \leq R \leq |z(t)| &\leq |z(t) \exp(P(z(t)))| \\ &\leq |z(t) \exp(Q(z(t)))|^k = R_*^k. \end{aligned}$$

On the other hand by assumption, we can take a natural number h and positive constants A^* and B^* such that

$$|A(z)| \leq A^*|z|^h, \quad |P(z) - Q(z)| \leq B^*|z|^n$$

for all values of z with $|z| \geq R^*$. Hence from (3.25),

$$|A(z \exp(P(z)))| \leq A^*R_*^{kh}, \quad |P(z) - Q(z)| \leq B^*R_*^{kn}$$

for all the points of C^* . Accordingly the functional equation (3.2) yields

$$(3.26) \quad |B(z \exp(Q(z)))| \leq A^*R_*^{kh} + B^*R_*^{kn}$$

on this curve C^* . It therefore follows from (3.24) and (3.26) that

$$|B(z)| \leq A^*|z|^{kh} + B^*|z|^{kn}$$

on the circle $|z| = R_* > R$. Consequently the function $B(z)$ is regular or has a pole at the point at infinity, since R may be chosen as large as we please and the quantities h, k, A^* and B^* are independent of R .

If $B(z)$ is regular at the infinity, then $B(z \exp(Q(z)))$ has a finite limit when z converges to infinity from the inside of the sector $|\arg z - c^*| \leq t^*$. Hence by (3.2) and (3.8),

$$\frac{A(z \exp(P(z)))}{z \exp(P(z))} \rightarrow 0$$

as z grows to infinity from the inside of this sector. Hereby $A(z)$ is also regular at the infinity. Thus from (3.2) again, $P(z) - Q(z)$ converges to some finite value as z grows to infinity. Accordingly $P(z) - Q(z)$ reduces to a constant function by Lemma 4.

Assume next that $B(z)$ has a pole of order q at the point at infinity. Then from (3.2) and (3.14),

$$(3.27) \quad \frac{A(z \exp(P(z)))}{(z \exp(Q(z)))^q} \rightarrow c_*$$

as z tends to infinity from the inside of the sector $|\arg z - c^*| \leq t^*$, where the limit c_* is finite and nonzero. Hence the function $A(z)$ has also a pole at the point at infinity. Let p be its order. Then from (3.27), the quantity

$$\frac{(z \exp(P(z)))^p}{(z \exp(Q(z)))^q} = z^{p-q} \exp(pP(z) - qQ(z))$$

has a finite and nonzero limit as z tends to infinity from the inside of this sector. Hereby $pP(z) - qQ(z)$ must be regular at the infinity and hence $p = q$. Consequently $P(z) - Q(z)$ must be constant again.

Secondly we consider the case $\cos(a^* - b^*) = -1$. For this case we can take two real numbers t_1 and t_2 such that

$$(3.28) \quad -1 < \cos(nt + b^*) < 0, \quad \sin(nt + b^*) > 0$$

for all real values of t with $t_1 \leq t \leq t_2$. We therefore take a positive number ε such that

$$(3.29) \quad b_* \cos(nt + b^*) < -2\varepsilon, \quad b^* \sin(nt + b^*) > \varepsilon$$

in the interval $t_1 \leq t \leq t_2$, and

$$(3.30) \quad \begin{aligned} b_* |\cos(nt_j + b^*) - \cos(nt_* + b^*)| &> 3\varepsilon, \\ b_* |\sin(nt_j + b^*) - \sin(nt_* + b^*)| &> 3\varepsilon, \end{aligned}$$

where $t_* = (t_1 + t_2)/2$ and $j=1, 2$. According to (3.1), for this positive ε , we can also take a positive number $R^* > 1$ such that

$$(3.31) \quad |Q(z) - bz^n| \leq \varepsilon |z|^n$$

for all values of z with $R^* \leq |z| < +\infty$. Here we may further assume that

$$(3.32) \quad \log x \leq \varepsilon x^n$$

for all real values of x with $x \geq R^*$. It then follows from (3.31) that

$$(3.33) \quad |\operatorname{Re} Q(re^{it}) - b_* r^n \cos(nt + b^*)| \leq \varepsilon r^n$$

for all real values of t and r with $r \geq R^*$. Hence from (3.29),

$$(3.34) \quad \operatorname{Re} Q(re^{it}) \leq -\varepsilon r^n$$

for $r \geq R^*$ and $t_1 \leq t \leq t_2$. This means that the function $z \exp(Q(z))$ converges uniformly to zero as z tends to infinity from the inside of the sector $t_1 \leq \arg z \leq t_2$.

Now let s and $u_j (j=1, 2)$ be points of the ring $R^* < |z| < +\infty$ satisfying $\arg s = t_*$, $\arg u_j = t_j$ and

$$(3.35) \quad |s \exp(Q(s))| = |u_j \exp(Q(u_j))|.$$

Then by appealing to (3.32), (3.33) and (3.35), we have

$$(3.36) \quad \begin{aligned} (b_* \cos(nt_* + b^*) - \varepsilon) s_*^n &\leq \log s_* + \operatorname{Re} Q(s) \\ &\leq (b_* \cos(nt_* + b^*) + 2\varepsilon) s_*^n, \end{aligned}$$

$$\begin{aligned}
(3.36) \quad & (b_* \cos(nt_j + b^*) - \varepsilon) v_j^n \leq \log s_* + \operatorname{Re} Q(s) \\
& \leq (b_* \cos(nt_j + b^*) + 2\varepsilon) v_j^n
\end{aligned}$$

with $s_* = |s|$ and $v_j = |u_j|$ ($j=1, 2$). Since $\cos(nt + b^*)$ is decreasing for $t_1 \leq t \leq t_2$ by (3.28), the condition (3.30) yields

$$\begin{aligned}
& b_* \cos(nt_1 + b^*) - b_* \cos(nt_* + b^*) > 3\varepsilon, \\
& b_* \cos(nt_* + b^*) - b_* \cos(nt_2 + b^*) > 3\varepsilon.
\end{aligned}$$

Hence from (3.36),

$$\begin{aligned}
& (b_* \cos(nt_1 + b^*) - \varepsilon) v_1^n \leq \log s_* + \operatorname{Re} Q(s) \\
& \leq (b_* \cos(nt_1 + b^*) - \varepsilon) s_*^n, \\
& (b_* \cos(nt_2 + b^*) + 2\varepsilon) s_*^n \leq \log s_* + \operatorname{Re} Q(s) \\
& \leq (b_* \cos(nt_2 + b^*) + 2\varepsilon) v_2^n.
\end{aligned}$$

Taking account of (3.29), we consequently have $v_1 \geq s_* \geq v_2$. It thus follows from (3.29), (3.30) and (3.31) that

$$\begin{aligned}
(3.37) \quad & \operatorname{Im} (Q(u_1) - Q(s)) \geq \varepsilon s_*^n, \\
& \operatorname{Im} (Q(u_2) - Q(s)) \leq -\varepsilon s_*^n.
\end{aligned}$$

By these facts, especially by (3.34) and (3.37), for all sufficiently large integers j , we can define curves $C_j: z = z_j(t)$ ($0 \leq t \leq 2\pi$) which satisfy

$$\begin{aligned}
(3.38) \quad & t_1 \leq \arg z_j(t) \leq t_2, \quad |z_j(t)| \geq j, \\
& z_j(t) \exp(Q(z_j(t))) = r_j \exp(it + it_*)
\end{aligned}$$

for $0 \leq t \leq 2\pi$, where r_j are positive constants. Evidently by virtue of (3.34), these constants r_j converge to zero as j goes to infinity. Here let us put these functions $z_j(t)$ into the functional equation (3.2). Then

$$(3.39) \quad P(z_j(t)) - Q(z_j(t)) = A(w_j(t)) - B(r_j \exp(it + it_*))$$

for $0 \leq t \leq 2\pi$, where $w_j(t) = z_j(t) \exp(P(z_j(t)))$. On the other hand from (3.29) and the assumption $\cos(a^* - b^*) = -1$, $\cos(nt + a^*) > 2\varepsilon/b_* > 0$ for $t_1 \leq t \leq t_2$. Hence

$$(3.40) \quad \frac{z^n}{z \exp(P(z))} \longrightarrow 0$$

uniformly as z converges to infinity from the inside of the sector $t_1 \leq \arg z \leq t_2$. Therefore from (3.38) and (3.39), we can take a positive integer k independent of j such that

$$(3.41) \quad |B(r_j \exp(it + it_*))| \leq |w_j(t)|^k$$

for $0 \leq t \leq 2\pi$. Furthermore from (3.34), it is also possible to find a positive integer h such that

$$(z \exp(P(z))) (z \exp(Q(z)))^h \longrightarrow 0$$

uniformly as z converges to infinity with $t_1 \leq \arg z \leq t_2$. Hereby from (3.38),

$$(3.42) \quad |w_j(t)| r_j^h \longrightarrow 0$$

as j becomes infinite. Since r_j converges to zero as j goes to infinity, we thus conclude from (3.41) and (3.42) that the function $B(z)$ is regular or has a pole at the origin.

If $B(z)$ is regular at the origin, then we can easily see that $A(z)$ is also regular at the origin. Hence from (3.2), (3.34) and (3.40), the function $P(z) - Q(z)$ has a finite limit when z tends to the infinity along the ray $\arg z = t_*$. However since $\cos(a^* - b^*) = -1$, $P(z) - Q(z)$ has a pole of order n at the point at infinity. This is untenable. Suppose next that the function $B(z)$ has a pole of order q at the origin. Then from (3.2) and (3.34),

$$(3.43) \quad (z \exp(Q(z)))^q A(z \exp(P(z))) \longrightarrow c'$$

uniformly as z tends to infinity with $t_1 \leq \arg z \leq t_2$, where the limit c' is finite and nonzero. Thereby from (3.43), $A(z)$ has a pole at the infinity. Consequently by means of (3.43), with a suitable positive integer p ,

$$(z \exp(Q(z)))^q (z \exp(P(z)))^p \longrightarrow c''$$

uniformly when z converges to infinity with $t_1 \leq \arg z \leq t_2$, where the limit c'' is also finite and nonzero. This yields that the function $pP(z) - qQ(z)$ must be regular at the infinity. However this is clearly impossible.

We have therefore obtained the following assertion.

LEMMA 5. *Suppose that the function $A(z)$ does not have an essential singularity at the point at infinity. Then the functions $P(z)$ and $Q(z)$ differ by a constant.*

A simple modification of the above argument yields the following result.

LEMMA 6. *Suppose that $A(z)$ is regular or has a pole at the origin. If $m=n$, then $P(z) - Q(z)$ reduces to a constant function.*

We are now in a position to piece together the foregoing lemmas and prove the next assertion.

LEMMA 7. *Let $P(z)$, $Q(z)$, $A(z)$ and $B(z)$ be the functions satisfying (3.1) and (3.2). Suppose that $n \geq 1$, $m \leq n$ and the function $P(z) - Q(z)$ is not constant. Then the entire function $A(e^z)$ satisfies*

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \log T(r, A(e^z)) > 0.$$

Proof. By virtue of Lemmas 2 and 4, with a suitable positive integer k , either the function $z^{-k}A(z)$ converges to zero as z approaches the point at infinity along some asymptotic curve or the function $z^kA(z)$ converges to zero as z approaches the origin along some asymptotic curve. Suppose first that the former case occurs. If

$A(z)$ does not have an essential singularity at the point at infinity, then the function $P(z) - Q(z)$ must be constant by Lemma 5. Hence $A(z)$ has an essential singularity at the infinity, so does $z^{-k}A(z)$. It therefore follows from Lemma 3 that the entire function $S(z) = \exp(-kz)A(e^z)$ must satisfy

$$(3.44) \quad \liminf_{r \rightarrow +\infty} \frac{1}{r} \log T(r, S(z)) > 0.$$

On the other hand it is clear that

$$T(r, S(z)) \leq O(r) + T(r, A(e^z))$$

for all real values of r . Thereby from (3.44), we at once obtain the desired result.

Next let us consider the latter case. If $m < n$, by what we have seen in the proof of Lemma 2, the former case necessarily occurs. Hence we may suppose that $m = n$. By virtue of Lemma 6, we may further suppose that the function $A(z)$ has an essential singularity at the origin. Then the function $z^{-k}A(1/z)$ has an essential singularity at the point at infinity and converges to the value zero when z tends to the point at infinity along some asymptotic curve. Accordingly by setting $S^*(z) = \exp(-kz)A(e^{-z})$, we obtain by use of Lemma 3 that

$$(3.45) \quad \liminf_{r \rightarrow +\infty} \frac{1}{r} \log T(r, S^*(z)) > 0.$$

Since $T(r, A(e^z)) = T(r, A(e^{-z}))$ and

$$T(r, S^*(z)) \leq O(r) + T(r, A(e^{-z}))$$

for all real values of r , we thereby obtain the desired result from (3.45). This completes the proof.

4. Proof of Theorem. Let $f(z)$ be a nonlinear entire function of finite lower order, and let $g(z)$ be a nonlinear entire function of exponential type. In what follows we assume that the functions

$f(z)-z$ and $g(z)-z$ are both periodic with the same period c' . Then there exist regular functions $A^*(z)$ and $P^*(z)$ in the punctured plane $0 < |z| < +\infty$ such that

$$(4.1) \quad f(z)-z = A^*(e^{cz}), \quad g(z)-z = P^*(e^{cz})$$

with the constant $c=2\pi i/c'$. Since $g(z)$ is of exponential type, so is $g(z)-z$. Hence $P^*(z)$ must be a rational function. Let us denote the composite function $f(g(z))$ with $F(z)$. Then the maximum modulus principle permits us to conclude that

$$M(r, F) \leq M(M(r, g), f),$$

and hence

$$(4.2) \quad T(r, F) \leq 3T(2M(r, g), f)$$

for all real values of r . Since $g(z)$ is of exponential type, it is possible to find a positive number ε such that

$$2M(r, g) \leq \exp(\varepsilon r)$$

for all sufficiently large values of r . Thereby (4.2) implies

$$\log T(r, F) \leq O(1) + \log T(\exp(\varepsilon r), f)$$

for all sufficiently large values of r . Since the lower order of $f(z)$ is finite by assumption, it thus follows that

$$(4.3) \quad \liminf_{r \rightarrow +\infty} \frac{1}{r} \log T(r, F) < +\infty.$$

Furthermore the function $F(z)$ satisfies the identity

$$(4.4) \quad F(z+c') - F(z) = c'.$$

Now let us suppose that the composite function $F(z)$ has another nontrivial factorization $f_*(g_*(z))$ with entire functions $f_*(z)$ and

$g_*(z)$. Since $F(z)$ satisfies the identity (4.4), by virtue of Proposition, the right factor $g_*(z)$ must satisfy the identity

$$(4.5) \quad g_*(z+c') - g_*(z) = d,$$

where d is a nonzero constant. By changing $g_*(z)$ to $c'g_*(z)/d$, if necessary, we can assume that this nonzero constant d is equal to the period c' . According to (4.4) and (4.5), the left factor $f_*(z)$ also satisfies the identity

$$(4.6) \quad f_*(z+c') - f_*(z) = c'.$$

Under these circumstances we want to show that the right factors $g(z)$ and $g_*(z)$ differ by a constant.

By means of the identities (4.5) and (4.6), the functions $f_*(z) - z$ and $g_*(z) - z$ are both periodic of period c' . Hence they can be written as

$$(4.7) \quad f_*(z) - z = B^*(e^{cz}), \quad g_*(z) - z = Q^*(e^{cz}),$$

where $B^*(z)$ and $Q^*(z)$ are regular functions in the annulus $0 < |z| < +\infty$. Of course, neither $B^*(z)$ nor $Q^*(z)$ is a constant function because the factorization $f_*(g_*(z))$ is not trivial. Therefore it is possible to find a positive number ε such that

$$(4.8) \quad T(r, f_*) \geq \varepsilon r, \quad T(r, g_*) \geq \varepsilon r$$

for all sufficiently large values of r . On the other hand by the maximum modulus principle again,

$$(4.9) \quad \begin{aligned} M(r, f_*(g_*(z))) &\geq M(R(r), f_*(z)), \\ R(r) &= AM(r/2, g_*) - (A+1)|g_*(0)| \end{aligned}$$

for all sufficiently large values of r , where A is the positive absolute constant. Since $F(z) = f_*(g_*(z))$, it thus follows that

$$(4.10) \quad \log T(2r, F) + O(1) \geq \log T(R(r), f_*),$$

so that from (4.8),

$$(4.11) \quad \log R(r) \leq \log T(2r, F) + O(1)$$

for all sufficiently large values of r . Here let us recall the condition (4.3). It then follows from (4.9) and (4.11) that

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \log M(r, g_*(z)) < +\infty.$$

Consequently taking account of (4.5), we can see that the function $g_*(z)$ must be of exponential type, so that the function $Q^*(z)$ must be rational. In addition to these facts, from (4.8) and (4.9), the quantity $R(r)$ satisfies

$$R(r) \geq A \exp(\varepsilon r/2) + O(1)$$

for all sufficiently large values of r . Hence the inequality (4.10) yields

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \log T(r, F) \geq \frac{\varepsilon}{4} \liminf_{r \rightarrow +\infty} \frac{\log T(r, f_*)}{\log r}.$$

Therefore from (4.3) again, the lower order of the left factor $f_*(z)$ must be finite.

Since $F(z) = f(g(z)) = f_*(g_*(z))$, using the representations (4.1) and (4.7), we obtain, setting $w = \exp(cz)$,

$$\begin{aligned} F(z) &= z + P^*(w) + A^*(w \exp(cP^*(w))) \\ &= z + Q^*(w) + B^*(w \exp(cQ^*(w))) \end{aligned}$$

in the whole plane. Hence the functions $A^*(z)$, $B^*(z)$, $P^*(z)$ and $Q^*(z)$ satisfy

$$(4.12) \quad P^*(z) + A^*(z \exp(cP^*(z))) = Q^*(z) + B^*(z \exp(cQ^*(z)))$$

in the annulus $0 < |z| < +\infty$. Setting $P(z) = cP^*(z)$, $Q(z) = cQ^*(z)$, $A(z) = -cA^*(z)$ and $B(z) = -cB^*(z)$, we therefore obtain the

functional equation

$$(4.13) \quad P(z) - Q(z) = A(ze^{P(z)}) - B(ze^{Q(z)}).$$

Assume now that the function $P^*(z) - Q^*(z)$ is not constant and either $P^*(z)$ or $Q^*(z)$ has a pole at the point at infinity. It then follows that $P(z) - Q(z)$ is not constant and $P(z)$ or $Q(z)$ has a pole at the infinity. In view of Lemma 7 and the symmetric property of (4.13), we then have either

$$(4.14) \quad \liminf_{r \rightarrow +\infty} \frac{1}{r} \log T(r, A(e^z)) > 0$$

or else

$$(4.15) \quad \liminf_{r \rightarrow +\infty} \frac{1}{r} \log T(r, B(e^z)) > 0.$$

If (4.14) holds, the entire function $A(e^z)$ is of infinite lower order. Hence the lower order of the entire function $A^*(e^z)$ is also infinite. Consequently by the representation (4.1), the function $f(z)$ has infinite lower order. This is a contradiction. Similarly if (4.15) holds, then the entire function $B^*(e^z)$ is of infinite lower order. Hence the function $f_*(z)$ has also infinite lower order. This is a contradiction again. Accordingly the functions $P^*(z)$ and $Q^*(z)$ must differ by a constant provided that $P^*(z)$ or $Q^*(z)$ has a pole at the point at infinity.

Suppose next that $P^*(z) - Q^*(z)$ is not constant and either $P^*(z)$ or $Q^*(z)$ has a pole at the origin. In this case we set $P(z) = -cP^*(1/z)$, $Q(z) = -cQ^*(1/z)$, $A(z) = cA^*(1/z)$ and $B(z) = cB^*(1/z)$. Then the functional equation (4.12) also becomes (4.13), and either the function $P(z)$ or $Q(z)$ has a pole at the point at infinity. Hence by the same fashion as just above, $A^*(e^{-z})$ or $B^*(e^{-z})$ has infinite lower order. Therefore either $f(z)$ or $f_*(z)$ is of infinite lower order. This is a contradiction by assumption. Consequently the function $P^*(z) - Q^*(z)$ must be constant if $P^*(z)$ or $Q^*(z)$ has a pole at the origin. Since $P^*(z)$ and $Q^*(z)$ are both rational, the difference $P^*(z) - Q^*(z)$ is always constant. Therefore by the representations

(4.1) and (4.7), the functions $g(z)$ and $g_*(z)$ differ by a constant. This is what we had to prove. We have therefore shown that the composite function $f(g(z))$ is uniquely factorizable relative to the family to entire functions.

To complete the proof we further need to show the unique factorizability of this composite function relative to the family of meromorphic functions. However this fact is an immediate consequence of the following final lemma.

LEMMA 8. *Let $A(z)$ and $B(z)$ be transcendental entire functions such that the composite function $A(B(z))$ is uniquely factorizable relative to the family of entire functions. If the function $A(B(z))$ is not uniquely factorizable relative to the family of meromorphic functions, then $A(z)$ or $A(B(z))$ is a periodic function.*

In fact by making use of (4.1) and (4.4), we can easily verify that neither $f(z)$ nor $f(g(z))$ is periodic. Hereby the unique factorizability of the composite function $f(g(z))$ follows from that of this function relative to the family of entire functions.

Henceforth we shall prove Lemma 8.

Proof of Lemma 8. By assumption, there exist meromorphic functions $C(z)$ and $D(z)$ satisfying $A(B(z)) = C(D(z))$. Of course, we may assume that $C(z)$ or $D(z)$ has poles. Assume first that $C(z)$ is entire and $D(z)$ has poles. Then $C(z)$ must be regular at the point at infinity. This is clearly impossible.

We now consider the case where $C(z)$ has poles and $D(z)$ is entire. Then the function $C(z)$ has exactly one pole at a point s , and the entire function $D(z)$ never takes the value s . Hence with some natural number n ,

$$C^*(z) = (z-s)^n C(z)$$

is entire with $C^*(s) \neq 0$, and with a suitable entire function $D^*(z)$, the function $D(z)$ can be written in the form

$$D(z) = s + \exp(D^*(z)).$$

Hereby the composite function $C(D(z))$ becomes

$$\begin{aligned} C(D(z)) &= (D(z) - s)^{-n} C^*(D(z)) \\ &= \exp(-nD^*(z)) C^*(s + \exp(D^*(z))). \end{aligned}$$

Here let us consider the function $E(z)$ defined with

$$E(z) = \exp(-nz) C^*(s + e^z).$$

Then this function $E(z)$ is surely entire and periodic. Further since $E(x) \exp(nx)$ tends to the nonzero value $C^*(s)$ as the real variable x becomes to negatively infinite, the function $E(z)$ does not reduce to a polynomial. Evidently $A(B(z)) = C(D(z)) = E(D^*(z))$. Hence by the assumption that $A(B(z))$ is uniquely factorizable in entire sense, $D^*(z)$ is linear or else $E(z) = A(az + b)$ with constants a and b . In the former case, the function $A(B(z))$ must be periodic by the periodicity of $E(z)$. In the latter case, the left factor $A(z)$ is necessarily periodic.

Finally suppose that neither $C(z)$ nor $D(z)$ is entire. Then the function $C(z)$ is regular at the point at infinity. Hence $C(z)$ reduces to a rational function and has at most two poles. For a moment we further suppose that $C(z)$ has only one pole at a point s . Then this rational function $C(z)$ is representable as

$$C(z) = (z - s)^{-n} P(z),$$

where n is a natural number and $P(z)$ is a polynomial of degree at most n with $P(s) \neq 0$. Since $D(z)$ fails to take the value s ,

$$D_*(z) = \frac{1}{D(z) - s}$$

must be entire. We thus obtain

$$\begin{aligned} C(D(z)) &= (D_*(z))^n P(D(z)) \\ &= \sum_{j=0}^k b_j (1 + sD_*(z))^j (D_*(z))^{n-j}, \end{aligned}$$

where $P(z) = b_0 + b_1 z + \dots + b_k z^k$ with $b_k \neq 0$. Setting

$$Q(z) = \sum_{j=0}^k b_j (1 + sz)^j z^{n-j},$$

we therefore obtain $A(B(z)) = C(D(z)) = Q(D_*(z))$. Since $0 \leq k \leq n$ and $P(s) \neq 0$, the degree of the polynomial $Q(z)$ is just n . Of course, the functions $Q(z)$ and $D_*(z)$ are both entire. Hence by assumption, $Q(z)$ must be linear. Hereby $n=1$, so that the function $C(z)$ reduces to a linear transformation. We may therefore reject this case. Consequently the rational function $C(z)$ has two distinct poles at points u and v . It then follows that

$$C(z) = (z-u)^{-m} (z-v)^{-n} P^*(z),$$

where m and n are natural numbers, $P^*(z)$ is a polynomial with $P^*(u)P^*(v) \neq 0$ and $\deg P^*(z) \leq m+n$. Furthermore since the function $D(z)$ never takes the values u and v , we can express $D(z)$ as

$$\frac{D(z) - u}{D(z) - v} = \exp(S(z))$$

with a suitable entire function $S(z)$. We thereby have

$$C(D(z)) = (u-v)^{-m-n} (1 - \exp(S(z)))^{m+n} \exp(-mS(z)) P^*(D(z)).$$

The last factor $P^*(D(z))$ becomes

$$P^*(D(z)) = \sum_{j=0}^k c_j (u - v \exp(S(z)))^j (1 - \exp(S(z)))^{-j},$$

where $P^*(z) = c_0 + c_1 z + \dots + c_k z^k$ with $c_k \neq 0$. Hence it follows that

$$C(D(z)) = (u-v)^{-m-n} \exp(-mS(z)) Q^*(\exp(S(z))),$$

$$Q^*(z) = \sum_{j=0}^k c_j (u - vz)^j (1 - z)^{m+n-j}.$$

By setting

$$E^*(z) = (u-v)^{-m-n} \exp(-mz) Q^*(e^z),$$

we finally obtain the identity $C(D(z)) = E^*(S(z))$. Evidently the entire function $E^*(z)$ is periodic. Since $Q^*(0) = P^*(u) \neq 0$, $E^*(z)$ is surely transcendental. Therefore either the function $S(z)$ is linear or $E^*(z) = A(az+b)$ with suitable constants a and b . Accordingly $A(z)$ or $A(B(z))$ is periodic. This completes the proof of Lemma 8.

References

- [1] A. Edrei, The deficiencies of meromorphic functions of finite lower order, Duke Math. J., 31 (1964), 1-22.
- [2] W. K. Hayman, Meromorphic functions, Oxford Univ. Press, 1964.
- [3] R. Nevanlinna, Eindeutige analytische Funktionen, 2nd ed., Berlin, 1953.
- [4] H. Urabe, Uniqueness of the factorization under composition of certain entire functions, J. Math. Kyoto Univ., 18 (1978), 95-120.