On certain real entire functions

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Let $f(z)$ be a nonconstant real entire function (i.e., $f(z)$ takes only real values on the real axis), and let $E(f)$ be the set of real numbers t for which all the roots of the equations $f(z)=t$ are real only. Suppose that the set $E(f)$ contains more than two real numbers. Then $T(r, f)=O(r)$, where $T(r, f)$ means the characteristic function of $f(z)$. Hence the function $f(z)$ has at most order one and mean type. Furthermore the set $E(f)$ is a closed interval in the real field and if this interval $E(f)$ is unbounded, then $f(z)$ reduces to a polynomial of degree at most two (cf. Edrei [2]).

The purpose of this paper is to show the following theorems on such real entire functions.

Theorem 1. *Let $f(z)$ be a nonconstant real entire function. Assume that the set $E(f)$ contains more than two points. Then the limit*

$$(*) \qquad \lim_{r\to\infty} \frac{T(r, f)}{r}$$

exists finitely.

Theorem 2. *Under the hypotheses of Theorem* 1, *assume further that the limit* (*) *is positive. Then the set $E(f')$ of the derivative $f'(z)$ also contains more than two points.*

1. **Preliminaries.** Let $f(z)$ be a nonconstant real entire function

whose set $E(f)$ contains more than two points. Let us set $g(z)=af(z)+b$ with real constants $a(\neq 0)$ and b. Then the function $g(z)$ is also real entire and $T(r,g)=T(r,f)+O(1)$. Furthermore it is clear by definition that if $E(f)=[u,v]$, then the set $E(g)$ coincides with the closed interval $[au+b, av+b]$. Hence we can assume without loss of generality that the set $E(f)$ is the closed interval between -1 and 1.

Let z^* be a point of the upper half plane H such that $f(z^*)$ is real and greater than one. Since the function $f(z)$ fails to take the values 1 and -1 in the upper half plane H, $(f(z))^2-1$ never vanishes there. Hence we can define the regular function $A(z)$ in the upper half plane H such that

$$(A(z))^2=(f(z))^2-1$$

and that $A(z^*)$ is real positive.

Here for a moment we assume that $\mathrm{Re}(\overline{f(w)}A(w))=0$ for some point w of the upper half plane H. Then $(\overline{f(w)}A(w))^2\leqq 0$, so that $(\overline{f(w)})^2((f(w))^2-1)\leqq 0$. It thus follows that $(\overline{f(w)})^2$ must be real and $0\leqq(f(w))^4\leqq(f(w))^2$. This means that the value $f(w)$ is real and satisfies $-1\leqq f(w)\leqq 1$. This is clearly impossible by hypotheses. Therefore $\mathrm{Re}(\overline{f(z)}A(z))$ is always positive or negative in the upper half phane H. Moreover, since $f(z^*)$ and $A(z^*)$ are both real and positive, it then follows that $\mathrm{Re}(\overline{f(z)}A(z))$ is positive at each point of the upper half plane H.

By the relation $(A(z))^2=(f(z))^2-1$, we have $|A(z)|^2+|f(z)|^2\geqq 1$ in the upper half plane H. Hereby

$$|f(z)+A(z)|^2=|f(z)|^2+2\mathrm{Re}(\overline{f(z)}A(z))+|A(z)|^2>1$$

at each point of H. In particular, $|f(z)+A(z)|>1$ in the upper half plane H. By this fact, we can further define the regular function

$$H(z)=i\log(f(z)+A(z))$$

in the upper half plane H, where the branch of the logarithm is chosen so that $H(z^*)$ is pure imaginary. Evidently

$$f(z)+A(z)=\exp(-iH(z))$$

by definition. It thus yields that

$$\begin{aligned}\exp(-2iH(z))&=(f(z))^2+2f(z)A(z)+(A(z))^2\\&=2(f(z))^2+2f(z)A(z)-1\\&=2f(z)\exp(-iH(z))-1.\end{aligned}$$

Consequently we deduce that

$$2f(z)=\exp(iH(z))+\exp(-iH(z)),$$

so that $f(z)=\cos H(z)$ in the upper half plane H.

Now by $V(z)$, let us denote the imaginary part of the function $H(z)$. Then the function $V(z)$ is harmonic and

$$V(z)=\log|f(z)+A(z)|$$

in the upper half plane H. Furthermore since $|f(z)+A(z)|>1$ in H, this function $V(z)$ is always positive there. It thus follows from a well known result of Caratheodory [1] that the quotient $H(z)/z$ converges to a finite value s uniformly as z tends to infinity from the inside of an arbitrarily fixed angular region $|\arg z-\pi/2|\leqq t^*<\pi/2$. Here the limit s is the quantity defined with

$$s=\inf_{y>0}\frac{\operatorname{Im} H(x+iy)}{y}=\inf_{y>0}\frac{V(x+iy)}{y},$$

so that this limit s is a real and nonnegative value. Of course, we have $V(x+iy)\geqq sy$ for values of $x+iy$ with $y>0$.

2. **Proof of Theorem 1.** Let $f(z)$ be a nonconstant real entire function whose set $E(f)$ contains more than two points. Then by what is mentioned before, we can suppose that the set $E(f)$ coincides with the closed interval between -1 and 1.

Let $H(z)$ be the regular function defined in the previous section.

Then $f(z)=\cos H(z)$ in the upper half plane H. Hence

$$2|f(z)|\leqq\exp(V(z))+\exp(-V(z))$$

at each point of H, where $V(z)$ stands for the imaginary part of the function $H(z)$. Since $V(z)$ is always positive in H, we thus obtain

$$\log|f(z)|\leqq\log^+|f(z)|\leqq V(z)$$

there. Furthermore since

$$\lim_{r\to+\infty}\frac{1}{r}H(re^{it})e^{-it}=s$$

for real values of t with $0<t<\pi$, and this limit s is real, we can find

$$\lim_{r\to+\infty}\frac{1}{r}V(re^{it})=s\sin t$$

for $0<t<\pi$. Hence it follows that

$$\limsup_{r\to+\infty}\frac{1}{r}\log^+|f(re^{it})|\leqq s\sin t$$

for $0<t<\pi$. On the other hand, since $T(r,f)=O(r)$, we have $\log M(r,f)=O(r)$ for real positive values of r, where $M(r,f)$ stands for the maximum of $|f(z)|$ on the circle $|z|=r$. Hence it is possible to find a positive number M satisfying

$$\log^+|f(re^{it})|\leqq Mr$$

for real values of r and t with $r>0$. Therefore for an arbitrary unbounded increasing sequence $\{r_n\}$ of positive numbers, it follows from Lebesgue's convergence theorem that

$$\limsup_{n\to\infty}\frac{1}{r_n}\int_0^\pi\log^+|f(r_ne^{it})|dt$$

$$\leqq \int_0^\pi \limsup_{n\to\infty} \frac{1}{r_n} \log^+ |f(r_n e^{it})| \, dt$$

$$\leqq \int_0^\pi s \sin t \, dt.$$

Since $\overline{f(z)} = f(\bar{z})$, we thus have

$$\limsup_{n\to\infty} \frac{T(r_n, f)}{r_n} \leqq \frac{2}{\pi} s,$$

so that we get finally

$$\limsup_{r\to+\infty} \frac{T(r, f)}{r} \leqq \frac{2}{\pi} s.$$

Again, let us recall the relation $f(z) = \cos H(z)$ and the inequality $V(z) \geqq sy$. Then it is clear that

$$\exp(V(z)) \leqq 1 + |f(z)|,$$

$$sy \leqq V(z) \leqq \log 2 + \log^+ |f(z)|$$

for values of $z = x + iy$ with $y > 0$. Hence we easily obtain

$$\liminf_{r\to+\infty} \frac{T(r, f)}{r} \geqq \frac{2}{\pi} s.$$

Consequently, $T(r, f)/r$ converges to the real and nonnegative value $2s/\pi$ as r tends to infinity. This completes the proof.

3. **Preliminary Lemmas.** Before proceeding with the proof of Theorem 2, we need two preliminary lemmas.

Let $f(z)$ be a nonconstant real entire function whose set $E(f)$ coincides with the closed interval $[-1, 1]$, and let $H(z)$ be the auxiliary function corresponding to this function $f(z)$ as in the section 1. The quotient $H(z)/z$ converges to the real value s as z tends to infinity from the inside of the upper half plane H.

LEMMA 1. *With the above notation*

$$s^2(f(x))^2+(f'(x))^2 \geqq s^2$$

for all real points x.

PROOF. Let $A(z)$ be the regular function defined in the section 1. Then by $(A(z))^2=(f(z))^2-1$, it is clear that the function $A(z)$ has continuous boundary values on the real axis and that $A(x)$ is pure imaginary for a real point x at which $|f(x)|<1$. Hence by the definition of $H(z)$, the function $H(z)$ has also continuous boundary values on the real axis and it takes real values for real points x with $|f(x)|<1$. It therefore follows from the principle of reflection that $H(z)$ is regular at each real point x such that $|f(x)|<1$. Hereby $H'(x)$ exists and is real, so that

$$H'(x)=\lim_{y\to+0}\frac{\operatorname{Im} H(x+iy)-\operatorname{Im} H(x)}{y}=\lim_{y\to+0}\frac{\operatorname{Im} H(x+iy)}{y}$$

at such a real point x. On the other hand, the imaginary part of the function $H(z)$, which we denote with $V(z)$, satisfies the inequality $V(x+iy)\geqq sy$ for real values of x and y with $y>0$. Hence we find $H'(x)\geqq s$ at each real point x provided that $|f(x)|<1$. Here let us recall the relation $f(z)=\cos H(z)$. Then $f'(z) = -H'(z)\sin H(z)$. Consequently we have that

$$s^2(f(x))^2+(f'(x))^2\geqq s^2\cos^2 H(x)+s^2\sin^2 H(x)=s^2$$

at every real point x with $|f(x)|<1$.

For a real point x at which $|f(x)|\geqq 1$, our desired inequality is trivial. Hereby Lemma 1 is proved.

LEMMA 2. *Suppose with the above notation that the limit value s is positive. Then the derivative $f'(z)$ has infinitely many positive real zeros and negative real zeros. Furthermore if $f''(x^*)=0$ at some real point x^*, then $|f'(x^*)|$ is not less than s.*

PROOF. Since $T(r, f)/r$ tends to the real value $2s/\pi$ as r becomes positively infinite by Theorem 1, and since the value s is not zero by assumption, the order of $f(z)$ is exactly one.

Assume now that the derivative $f'(z)$ has a finite number of positive zeros. Then the function $f(x)$ is strictly increasing or decreasing for sufficiently large real x. Hence we can find a real number c so that $f(x)$ does not take the values 1 and -1 for $x \geqq c$. Here let us consider the function $F(z)$ defined with $f(c-z^2)$. Then it is clear that $F(z)$ is also real entire and all the roots of the two equations $F(z)=1$ and $F(z)=-1$ are distributed on the real axis only. So it yields that $T(r, F)=O(r)$. On the other hand it follows from definition that $T(r^2, f)=O(T(r, F))$. Therefore we have $T(r, f)=O(r^{1/2})$. This is a contradiction. Hence the function $f'(z)$ has infinitely many positive zeros. Similarly, $f'(z)$ has infinitely many negative zeros.

Let x^* be a real point at which $f''(x^*)=0$. Assume for a moment that $f'(x^*)$ is also zero. Since the order of $f(z)$ is one, we have

$$\frac{d}{dz}\left(\frac{f'(z)}{f(z)}\right)=\sum_j \frac{-1}{(z-a_j)^2}$$

for values of z with $z \neq a_j$, where a_j stand for the zeros of $f(z)$. Hence if $f(x^*) \neq 0$, then $(f'(z)/f(z))'$ is surely negative at the point x^*. However this is clearly absurd. Hereby $f(x^*)$ must be zero. This is absurd again. It thus follows that $f'(z)$ does not vanish at this real point x^*.

Let a and b be successive real zeros of $f'(z)$ such that $a<x^*<b$. Since the set $E(f)$ is the closed interval between -1 and 1, the values $f(a)$ and $f(b)$ satisfy $|f(a)| \geqq 1$ and $|f(b)| \geqq 1$. Now we can claim that $f(x)$ has a zero point in the interval (a, b). For otherwise $f'(x)/f(x)$ decreases strictly for $a \leqq x \leqq b$, but it vanishes at both points a and b. It therefore follows from Lemma 1 that the maximum value of $|f'(x)|$ on this interval (a, b) must be greater than the limit value s. On the other hand, since the order of $f'(z)$ is at most one and all the zeros of $f'(z)$ are real only, $f''(x)/f'(x)$ also strictly decreases in the open interval (a, b). In particular $f''(x)/f'(x)$ vanishes at most once in this interval. Therefore for $a \leqq x \leqq b$, $|f'(x)|$ assumes its maximum value at the point x^*. Accordingly we find that $|f'(x^*)| \geqq s$, which is to be proved. This completes the proof.

4. **Proof of Theorem 2.** Let $f(z)$ be a nonconstant real entire function satisfying the hypotheses of Theorem 2. As before we can assume that the set $E(f)$ is equal to the closed interval between -1 and 1. Hence we can make use of the notations and the results which we state in the previous sections.

Hereafter we want to claim that the derivative $f'(z)$ never takes the real value s in the upper half plane H, where s is the quantity in Lemmas 1 and 2. Of course, s is positive by assumption.

On the contrary we suppose that there exists a point z^* in the upper half plane H at which $f'(z^*)=s$. Since $f''(z)$ takes no zeros in H, $f''(z^*)$ is not equal to zero. Hence we can consider the regular element of the inverse function of $f'(z)$ with center s and mapping the point s to the point z^*. Now let us continue analytically this regular element along the real axis from the point s toward the origin. Then by this continuation, we can define the curve C : $z=z(t)$ $(0\leqq t<t^*)$ satisfying $z(0)=z^*$ and $f'(z(t))=s-t$ for $0\leqq t<t^*$. Here let us notice that the initial point z^* is a point of the upper half plane H and that the function $f'(z)$ takes real values only on this curve C and the real axis. Hence if the curve C intersects the real axis at some point v, then $f''(v)$ must be zero. So $|f'(v)|\geqq s$ by virtue of Lemma 2. Furthermore let us notice that all the zeros of $f'(z)$ are real only. Then we can at once conclude that this continuation never continues to the origin. Therefore the continuation defines a transcendental singularity at some real point u with $0\leqq u<s$. Consequently we may assume that the curve $C: z=z(t)$ $(0\leqq t<t^*)$ is an asymptotic curve which starts from the point z^* and satisfies $f'(z(t))=s-t$ for $0\leqq t<t^*$, where $t^*=s-u$. Of course, the curve C never intersects the real axis, so that it is contained entirely in the upper half plane H. Since $f'(z)$ is of order one and has real zeros only, the imaginary part of the logarithmic derivative $f''(z)/f'(z)$ is always negative in the upper half plane H. Hence the imaginary part of $f''(z(t))$ is surely negative for $0\leqq t<t^*$, because $f'(z(t))$ is real and positive there. On the other hand by construction, the curve C is an analytic one. Therefore $f''(z(t))z'(t)=-1$ for $0\leqq t<t^*$. Hereby it follows that the imaginary part of $z'(t)$ is always negative, so that of $z(t)$ is strictly decreasing for $0\leqq t<t^*$. This means that the asymptotic curve C lies entirely in the strip $0<\operatorname{Im} z\leqq \operatorname{Im} z^*$. Since C approaches the point at infinity, the real part of $z(t)$ therefore becomes positively infinite or negatively infinite as the real parameter t tends to t^*. We may assume for definiteness that the former case occurs. Then it is possible to take a real number r such that the line $\mathrm{Re}\, z=r$ never crosses the curve C. Let C^* be the curve which is symmetric to the curve C with respect to the real axis. Then by assumption, the curves C and C^* are both contained entirely in the half strip $\mathrm{Re}\, z \geqq r, |\mathrm{Im}\, z| \leqq \mathrm{Im}\, z^*$. By D, let us denote the unbounded subregion of this half strip bounded by the lines $\mathrm{Re}\, z=r$, $\mathrm{Im}\, z=\pm \mathrm{Im}\, z^*$ and the curves C and C^*. Of course, by construction, the region D is simply connected and is symmetric with respect to the real axis. Furthermore D contains all sufficiently large real points. Now the function $f'(z)$ takes real values on the curve C and it converges to the real value u as z tends to infinity along C. Hereby since $f'(z)$ is also real entire, $f'(z)$ also converges to the value u as z tends to infinity along the curve C^*. In particular, $f'(z)$ is bounded on the boundary of D. Here let us remark that $f'(z)$ satisfies $\log |f'(z)|=O(|z|)$. It then follows from the maximum principle that the function $f'(z)$ is also bounded in the region D, so that $f'(z)$ approaches the real value u as z tends to infinity from the inside of D. Especially $f'(z)$ tends to the value u as z does to infinity along the positive real axis. However this is clearly impossible by Lemma 2. Consequently the function $f'(z)$ never assumes the value s in the upper half plane H. Since $f'(z)$ is real for real z, the equation $f'(z)=s$ has only real roots. Using the exactly same argument as above, we can also show that all the roots of the equation $f'(z)=-s$ are distributed on the real axis only. It therefore follows that the set $E(f')$ of the function $f'(z)$ contains the closed interval between $-s$ and s. This completes the proof.

References

[1] L. Ahlfors, Conformal invariants, McGraw-Hill, New York, 1973.

[2] A. Edrei, Meromorphic functions with three radially distributed values, Trans. Amer. Math. Soc., 78 (1955), 276-293.