Distribution of values of entire functions of finite lower order

By Tadashi Kobayashi

It is well known that a simple geometrical restriction on the distribution of the values of an entire function is almost enough to bound the growth of the function. For instance, the following facts were proved ([2], [5], [6]).

A. *Let $f(z)$ be an entire function having only real zeros and real ones. Then $f(z)$ has at most order one and mean type.*

B. *Let $f(z)$ be an entire function of finite lower order. If all its zeros and all its ones lie in the strip $|\mathrm{Im}\, z| \leqq 1$, then $f(z)$ has at most order one and mean type.*

C. *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{w_n\}$ of complex numbers such that all the roots of the equations $f(z)=w_n$ are real. Then the function $f(z)$ reduces to a polynomial of degree at most two.*

D. *Let $f(z)$ be an entire function and let $T(r,f)$ be its characteristic function. Assume that there exists an unbounded sequence $\{w_n\}$ of complex numbers such that all the solutions of the equations $f(z)=w_n$ lie in some closed half plane. Assume further that*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r}=0.$$

Then the function $f(z)$ is a polynomial of degree not greater than two.

The purpose of this article is to investigate the possibility of proving analogous results in this direction. In what follows we assume acquaintance with the standard terminology and the fundamental concepts of Nevanlinna theory, and we use them without further introduction.

1. Entire functions of genus at most one

Let $f(z)$ be an entire function of genus at most one. Then its logarithmic derivative can be represented in the form

$$\frac{f'(z)}{f(z)}=\frac{m}{z}+C+\sum_{a_n\neq 0}\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right), \tag{1.1}$$

where a_n are the zeros of $f(z)$, m is a non-negative integer and C is a suitable complex constant.

Hereafter we suppose that all the zeros a_n satisfy

$$-1\leqq \operatorname{Re} a_n \leqq 1. \tag{1.2}$$

Then it follows from (1.1) and (1.2) that

$$\operatorname{Re}\frac{f'(z)}{f(z)}=C^*+\sum_n \operatorname{Re}\left(\frac{1}{z-a_n}\right) \tag{1.3}$$

with a suitable real constant C^*.

1.1 First of all we prove the following fact.

LEMMA 1. *Suppose that $f(z)$ satisfies*

$$\operatorname{Re}\frac{f'(z)}{f(z)}>0 \tag{1.4}$$

for values of $z=x+iy$ with $x>1$, and that $f(z)$ approaches infinity as the variable z tends to infinity along the positive real axis. Then for *an arbitrary point* $z^*=x^*+iy^*$ *with* $x^*>1$, *the function* $f(z)$ *assumes in the half plane* $x>x^*$ *all finite values* w *such that* $\arg w=\arg f(z^*)$ *and* $|w|>|f(z^*)|$.

Proof. By the condition (1.4), $f'(z)\neq 0$ for values of $z=x+iy$ with $x>1$. In particular, $f'(z^*)\neq 0$. Let w^* be $f(z^*)$, and let $E(w, w^*)$ stand for the regular element of the inverse function of $f(z)$ with center w^* and satisfying $E(w^*, w^*)=z^*$. Now let us continue analytically this element $E(w, w^*)$ along the ray I: $\arg z=\arg w^*$ toward the point at infinity. Then we have an analytic continuation $f^{-1}(I_s)$ with algebraic character along this ray I up to some point sw^* of I, where s is real finite greater than 1 or infinite. Of course, we may assume that the continuation $f^{-1}(I_s)$ defines a transcendental singularity at this end point sw^*, because $f(z)$ being an entire function does not assume the value infinite. From this continuation $f^{-1}(I_s)$, we can thus define the curve $C: z=z(t)$ $(1\leqq t<s)$ that goes from $z^*=z(1)$ to the point at infinity and satisfies

$$f(z(t))=tf(z^*) \tag{1.5}$$

for $1\leqq t<s$. Evidently, $\operatorname{Re} z(t)>1$ on some interval $1\leqq t\leqq t^*$ with $1<t^*<s$. Since $f'(x+iy)\neq 0$ for $x>1$, the function $z(t)$ is surely differentiable on this interval $1\leqq t\leqq t^*$. It therefore follows from (1.5) that

$$f'(z(t))z'(t)=f(z^*),$$

so that

$$\frac{f'(z(t))}{f(z(t))}=\frac{1}{tz'(t)}$$

for $1\leqq t\leqq t^*$. Hence by virtue of (1.4), the real part of $z(t)$ strictly increases as the real parameter t varies from 1 to t^*. Particularly, $\operatorname{Re} z(t)>x^*$ for $1<t\leqq t^*$. By this fact, we finally see that the asymptotic path C lies entirely in the half plane $x>x^*$ save for its initial point z^*. Consequently if the value s is infinite, then by means of (1.5), the function $f(z)$ assumes on the curve C all finite values w such that arg $w=\arg w^*$ and $|w|>|w^*|$. This is the desired result. Next let us suppose that the value s is finite. Then by (1.5) again, $f(z)$ converges to the finite value sw^* when z goes to infinity along the curve C. On the other hand by the assumption, $f(z)$ approaches infinity along the positive real axis. Therefore by means of the classical Lindelöf's theorem [8], the function $f(z)$ assumes every finite values except 0 in the half plane $x>x^*$. Hereby we also obtain the desired result. This completes the proof.

1.2 Now let us suppose that the real constant C^*, given by (1.3), is positive. Then it is clear from (1.3) that

$$\text{(1.6)} \qquad \operatorname{Re} \frac{f'(z)}{f(z)} \geqq C^* > 0$$

for values of $z=x+iy$ with $x>1$. Hence it is also clear from this (1.6) that

$$\log|f(x)|-\log|f(x_*)| = \int_{x_*}^{x} \operatorname{Re} \frac{f'(t)}{f(t)} dt$$

$$\geqq C^*(x-x_*)$$

for real values of x and x_* with $x>x_*>1$. Hereby $f(x)$ converges to infinity as the real variable x increases without bound. Therefore $f(z)$ satisfies the hypothesis of the above Lemma 1. Besides, by virtue of (1.6), we can find that

$$\begin{aligned} &\arg f(h+it)-\arg f(h+is) \\ &= \int_{s}^{t} \operatorname{Re} \frac{f'(h+iy)}{f(h+iy)} dy \\ &\geqq C^*(t-s) \end{aligned}$$

for real values of h, s and t with $h>1$ and $s<t$. This means that for each real h with $h>1$, the argument of $f(h+iy)$ is a continuous, strictly increasing and unbounded function of the real variable y. Accordingly we at once obtain the following

Lemma 2. *Let C^* be the real constant defined by* (1.3), *and let h be a real number greater than one. Suppose that the constant C^* is positive. Then in the half plane* $\operatorname{Re} z>h$, *the function $f(z)$ assumes all finite values w of some annulus of the form $R<|w|<+\infty$.*

Considering the function $f(-z)$ instead of the original $f(z)$, we can treat the case when the real constant C^* is negative.

Lemma 3. *Assume that C^* is negative. Then for each real h with $h>1$, the region $\{f(z):\operatorname{Re} z<-h\}$ covers some annulus $R<|w|<+\infty$ entirely.*

It remains to discuss the case where C^* is equal to 0. In this case we further suppose that $f(z)$ has at least one zero point. It then follows from the relation (1.3) that

$$\operatorname{Re}\frac{f'(z)}{f(z)}=\sum_n \operatorname{Re}\left(\frac{1}{z-a_n}\right), \tag{1.7}$$

so that

$$\log|f(x)|-\log|f(x_*)|=\sum_n \log\left|\frac{x-a_n}{x_*-a_n}\right|$$

for real values of x and x_* with $x>x_*>1$. Hereby $f(x)$ converges to infinity as the real variable x increases without bound. Hence in this case the function $f(z)$ also satisfies the assumption of Lemma 1 provided $f(z)$ has at least one zero point. In addition to this fact, by means of (1.7),

$$\begin{aligned} &\arg f(h+ir)-\arg f(h-ir) \\ &=\sum_n \int_{-r}^{r} \operatorname{Re}\left(\frac{1}{h+iy-a_n}\right)dy \end{aligned} \tag{1.8}$$

for positive values of h and r with $h>1$. Here let us recall the integral

$$\int_{-\infty}^{+\infty} \frac{x}{x^2+y^2} dy = \pi,$$

where x is an arbitrary real positive constant. Then from (1.8), we deduce that

$$\liminf_{r \to +\infty} (\arg f(h+ir) - \arg f(h-ir)) \geqq n\pi,$$

provided that $f(z)$ has at least n zeros counting multiplicity. Consequently if $f(z)$ has at least three zeros, then it is possible to find a positive number r^* satisfying

$$\arg f(h+ir^*) - \arg f(h-ir^*) = 2\pi.$$

By this observation, we obtain the following result.

LEMMA 4. *Assume that $f(z)$ has at least three zeros counting multiplicity, and that the real constant C^* is equal to zero. Then for each real h with $h>1$, the regions $\{f(z): \mathrm{Re}\, z>h\}$ and $\{f(z): \mathrm{Re}\, z<-h\}$ both contain some annulus of the form $R<|w|<+\infty$.*

1.3 The foregoing lemmas are sufficient to yield the following theorem.

THEOREM 1. *Let $f(z)$ be an entire function of genus at most one, all of whose zeros lie in the closed strip $-1 \leqq \mathrm{Re}\, z \leqq 1$. Assume that neither the region $\{f(z): \mathrm{Re}\, z>1\}$ nor the region $\{f(z): \mathrm{Re}\, z<-1\}$ contains any annulus of the form $R<|w|<+\infty$. Then the function $f(z)$ reduces to a polynomial of degree at most two.*

PROOF. By the above lemmas, the real constant C^* must be equal to 0. Furthermore $f(z)$ has at most two zeros. Hence this function $f(z)$ can be written in the form

$$f(z) = P(z) \exp(Az+B),$$

where A and B are suitable complex constants with $\mathrm{Re}\, A=0$, and $P(z)$

is a polynomial of degree at most two. For a moment suppose that $A \neq 0$. Then it is clear that $f(z)$ converges to 0 as z tends to infinity along the ray $\arg z = \pi/4$ or along the ray $\arg z = -\pi/4$. Hereby in the half plane $\operatorname{Re} z > 1$, the function $f(z)$ assumes all finite values infinitely many times other than 0. This is clearly absurd by the assumption. Therefore the constant A must be 0. This completes the proof.

Combining this theorem with the proposition B mentioned in the introduction, we can obtain the following fact.

COROLLARY. *Let $f(z)$ be an entire function of finite lower order. Suppose that there exists an unbounded sequence $\{w_n\}$ such that all the roots of the equations $f(z) = w_n$ lie in the closed strip $-1 \leqq \operatorname{Re} z \leqq 1$. Then $f(z)$ is a polynomial of degree not greater than two.*

This corollary is not new, since it is a main result of S. Kimura [4]. However, our deduction of this fact may be of some interest because of its simplicity.

2. Relations between growth and distribution of values

Let $f(z)$ be an entire funciton with $f(0) = 1$. Then for positive real values of r,

$$\begin{aligned}(2.1)\qquad &\frac{1}{\pi}\int_0^{2\pi} \log|f(re^{it})| \cos t \, dt \\ &= r \operatorname{Re}(f'(0)) + \sum_{|a_n|<r} \operatorname{Re}\left(\frac{r}{a_n} - \frac{\bar{a}_n}{r}\right),\end{aligned}$$

where a_n stand for the zeros of $f(z)$. Here we assume that all the zeros a_n are distributed in some angular region $|\arg z| \leqq \alpha < \pi/2$. It thus follows from this (2.1) that

$$\frac{1}{\pi}\int_0^{2\pi} |\log|f(re^{it})|| \, dt$$

$$\geqq Cr+(2N(r,0,f)+S(r)) \cos\alpha,$$

where $C=\mathrm{Re}\,(f'(0))$ and

$$S(r)=\int_0^r \left(\frac{r}{t^2}-\frac{1}{r}\right)N(t,0,f)\,dt. \tag{2.2}$$

Hence we find that

$$4T(r,f)\geqq Cr+2(1+\cos\alpha)N(r,0,f)+S(r)\cos\alpha \tag{2.3}$$

for positive real values of r.

2.1 Now we further assume that the order or the lower order of the function $N(r,0,f)$ is equal to 1. Then we have a sequence of Polya peaks of the second kind, order 1 for $N(r,0,f)$ [9]. In other words, there exist positive sequences $\{r_n\}$ and $\{s_n\}$ such that

$$r_n \longrightarrow +\infty, \quad s_n \longrightarrow +\infty, \quad s_n/r_n \longrightarrow 0 \tag{2.4}$$

as n approaches infinity, and such that

$$N(t,0,f)\geqq(1+o(1))\frac{t}{r_n}N(r_n,0,f) \tag{2.5}$$

for $s_n\leqq t\leqq r_n$, $n\geqq n^*$. Combining this (2.5) with (2.2), we thus obtain

$$S(r_n)\geqq(1+o(1))\left(\log\frac{r_n}{s_n}-\frac{1}{2}\right)N(r_n,0,f),$$

so that

$$\limsup_{r\to\infty}\frac{S(r)}{N(r,0,f)}=+\infty$$

by means of (2.4). Consequently if

$$\liminf_{r\to\infty} \frac{N(r, 0, f)}{r} > 0,$$

then the above (2.3) implies

$$\liminf_{r\to\infty} \frac{N(r, 0, f)}{T(r, f)} = 0.$$

Lemma 5. *Let $f(z)$ be an entire function of lower order one. Suppose that all but a finite number of its zeros lie in some angular region $|\arg z| \leqq \alpha < \pi/2$, and that*

$$(2.6) \qquad 0 < \liminf_{r\to\infty} \frac{T(r, f)}{r} \leqq +\infty.$$

Then

$$(2.7) \qquad \liminf_{r\to\infty} \frac{N(r, 0, f)}{T(r, f)} = 0.$$

Furthermore if the order of $f(z)$ is finite, then the function $f(z)$ has no finite deficient values other than 0.

Proof. Let $P(z)$ be a polynomial and let us set $F(z) = f(z)/P(z)$. Then it is clear that

$$T(r, F) - T(r, f) = O(\log r),$$

$$N(r, 0, F) - N(r, 0, f) = O(\log r).$$

Hence we may assume without loss of generality that all the zeros of $f(z)$ belong to the angular region $|\arg z| \leqq \alpha$ and $f(0) = 1$. Here we may further assume that

$$\liminf_{r\to\infty} \frac{N(r, 0, f)}{r} > 0.$$

For otherwise, the desired (2.7) follows at once by the assumption (2.6). Of course, the lower order of $N(r, 0, f)$ is not greater than that of $T(r, f)$. Thus the lower order of $N(r, 0, f)$ is equal to 1. Therefore by what mentioned just above, we obtain the desired result (2.7).

The second statement is an immediate consequence of this (2.7) and the second main theorem of Nevanlinna theory. This completes the proof.

2.2 Let $f(z)$ be an entire function of order at most one. We assume in what follows that there exist three real numbers s_1, s_2, s_3 and three distinct complex numbers w_1, w_2, w_3 such that

$$s_1 < s_2 < s_3 < s_1 + \pi, \tag{2.8}$$

and such that for each number $n (n=1, 2, 3)$, all the roots of the equation $f(z) = w_n$ lie in the closed strip

$$S_n = \{z : -1 \leqq \operatorname{Re}(z \exp(is_n)) \leqq 1\}. \tag{2.9}$$

Evidently, any function of the form $a + b\exp(cz)$, where a, b and c are constants, does not satisfy the above assumption. Hence the function $f(z)$ takes every finite values at least once.

Now let us consider the functions defined by

$$F_n(z) = f(u_n z) - w_n, \quad u_n = \exp(-is_n) \qquad (n=1, 2, 3). \tag{2.10}$$

Then these functions are entire and by the assumption, all the zeros of $F_n(z)$ are distributed in the closed strip $-1 \leqq \operatorname{Re} z \leqq 1$. Of course, the genera of the functions $F_n(z)$ are at most one. Hence by the definition (2.10), the function $f(z)$ approaches infinity along the ray $\arg z = -s_n$ or along the ray $\arg z = -s_n + \pi (n=1, 2, 3)$.

Lemma 6. (I) *Assume that $f(z)$ approaches infinity as z tends to infinity along the ray* $\arg z = -s_1$. *If $s_1 + \pi/2 < s_3$, then all but a finite number of the solutions of the equation $f(z) = w_2$ lie in the closed half strip*

$$S_2^- = \{z : |\operatorname{Re}(\bar{u}_2 z)| \leqq 1, \operatorname{Im}(\bar{u}_2 z) \leqq 0\}.$$

If $s_1+\pi/2 \geqq s_3$, *then all except a finite number of the roots of* $f(z)=w_3$ *are contained in the closed half strip*

$$S_3^- = \{z: |\mathrm{Re}(\bar{u}_3 z)| \leqq 1, \mathrm{Im}(\bar{u}_3 z) \leqq 0\}.$$

(II) *Assume that* $f(z)$ *approaches infinity along the ray* $\arg z = -s_1+\pi$. *Then either all but a finite number of the roots of the equation* $f(z)=w_2$ *lie in the closed half strip*

$$S_2^+ = \{z: |\mathrm{Re}(\bar{u}_2 z)| \leqq 1, \mathrm{Im}(\bar{u}_2 z) \geqq 0\},$$

or else all except a finite number of the roots of $f(z)=w_3$ *are contained in the closed half strip*

$$S_3^+ = \{z: |\mathrm{Re}(\bar{u}_3 z)| \leqq 1, \mathrm{Im}(\bar{u}_3 z) \geqq 0\}.$$

Proof. (I) We assume first that $s_1+\pi/2 < s_3$. Then it is possible to take two real numbers a and b such that

$$\begin{aligned} &-s_3+\pi/2 < a < -s_1 < b < -s_1+\pi/2, \\ &a < -s_2+\pi/2 < b. \end{aligned} \tag{2.11}$$

From (2.8) and (2.11), we can further take a positive number R such that the closed sector

$$H = \{z: a \leqq \arg z \leqq b, |z| \geqq R\}$$

is contained entirely in the intersection of the half planes $\mathrm{Re}(\bar{u}_1 z) > 1$ and $\mathrm{Re}(\bar{u}_3 z) < -1$. Hence by the assumption, in this sector H, the function $f(z)$ fails to take the two finite values w_1 and w_3. Besides, from (2.11), the ray $z = u_1 t (t \geqq R)$ lies in the sector H. It therefore follows from Lindelöf's theorem that

$$\lim_{r \to +\infty} f(re^{it}) = \infty$$

uniformly for $a^* \leqq t \leqq b^*$, where a^* and b^* are arbitrarily fixed real numbers with $a < a^* < b^* < b$. Here let us notice (2.11) again. Then we find that

$$\lim_{y\to+\infty} f(u_2(x+iy))=\infty$$

uniformly for $-1\leqq x\leqq 1$. In particular, there exists a positive real number R^* such that $|f(z)|>|w_2|$ provided $|\mathrm{Re}(\bar{u}_2z)|\leqq 1$ and $\mathrm{Im}(\bar{u}_2z)\geqq R^*$. By this fact, we at once have the desired result.

Next let us consider the case when $s_1+\pi/2\geqq s_3$. In this case, $-s_1<-s_2+\pi/2$ and $-s_1-\pi/2<-s_3+\pi/2$ by virtue of (2.8). Hence it is possible to find two real numbers c and d satisfying

$$\begin{gathered} -s_1-\pi/2<c<-s_1<d<-s_2+\pi/2, \\ c<-s_3+\pi/2<d. \end{gathered} \tag{2.12}$$

Let us set the sector $A: c\leqq \arg z\leqq d$. Then the intersection of this sector A and the closed strip S_1 defined by (2.9) is clearly bounded. Also, the intersection of A and S_2 is bounded. Therefore by the assumption, neither w_1 nor w_2 is taken infinitely often by $f(z)$ in the sector A. Moreover, by means of (2.12), the sector A contains the rays $\arg z=-s_1$ and $\arg z=-s_3+\pi/2$. Thus, using Lindelöf's theorem again, we can see that

$$\lim_{y\to+\infty} f(u_3(x+iy))=\infty$$

uniformly for $-1\leqq x\leqq 1$. This means that the equation $f(z)=w_3$ has only a finite number of solutions in the closed half strip S_3^+. This is the desired result. The proof of the assertion (I) is now complete.

(II) Let us consider the auxiliary function $F(z)=f(-z)$. Then it is clear that $F(z)$ is entire and of order at most one. Evidently, for each number n $(n=1, 2, 3)$, all the roots of the equation $F(z)=w_n$ lie in the closed strip S_n defined by (2.9). Furthermore $F(z)$ tends to infinity along the ray $\arg z=-s_1$. Hence by virtue of the above assertion (I), either all but a finite number of the roots of $F(z)=w_2$ are contained in S_2^-, or else all but a finite number of the roots of $F(z)=w_3$ lie in S_3^-. Turning back the original function $f(z)$ we at once obtain the assertion (II). The proof of Lemma 6 is now complete.

2.3 We are now in a position to prove the following theorem.

THEOREM 2. *Let $f(z)$ be an entire function of order at most one. Assume that there exist three different finite complex numbers w_1, w_2, w_3 and three closed strips S_1^*, S_2^*, S_3^* of the complex plane such that for each n $(n=1, 2, 3)$, all the roots of the equation $f(z)=w_n$ are distributed in the strip S_n^*. Assume further that no two of the three strips S_n^* run parallel with each other. Then the function $f(z)$ reduces to a polynomial.*

PROOF. By the assumption, we can assume without loss of generality that the strips S_n^* are defined by

$$S_n^*=\{z:-1\leqq \operatorname{Re}(z\exp(is_n))\leqq 1\},$$

where s_n are suitable real numbers with

$$s_1<s_2<s_3<s_1+\pi.$$

In what follows, as usual, by $f^{-1}(w_n)$, we denote the set of all the roots of the equation $f(z)=w_n$. Then by the above Lemma 6, for each pair p and q with $1\leqq p<q\leqq 3$, either the set $f^{-1}(w_p)$ or the set $f^{-1}(w_q)$ is contained in some closed half strip of the complex plane.

Here for a moment, we further assume that

$$\liminf_{r\to\infty}\frac{T(r,f)}{r}>0. \tag{2.13}$$

Then the genera of the functions $f(z)-w_n$ are exactly one. Hence the function $f(z)$ has the value w_n as a deficient value provided that all the roots of $f(z)=w_n$ lie in some closed half strip. Accordingly, at least two of the three values w_1, w_2, w_3 are deficient values of $f(z)$. However this is clearly absurd by Lemma 5. Consequently, (2.13) is false. It thus follows that all the zeros of $f'(z)$ are contained in each strip S_n^*, so that they are contained in the intersection of the strips S_1^*, S_2^* and S_3^*. Therefore $f'(z)$ has only a finite number of zeros. Hereby $f'(z)$ is a polynomial, since (2.13) is false. This is the desired result.

3. Further relations between growth and distribution of values

Let s be a real number with $0<s<\pi/2$, and let G_s stand for the angular region $|\arg(z-1)|<s$. Then the transformation

$$z=h(w)=1+\left(\frac{1-w}{1+w}\right)^{\beta} \qquad (\beta=2s/\pi) \tag{3.1}$$

maps conformlly the unit disk $|w|<1$ onto the region G_s. It is clear that

$$w=h^{-1}(z)=\frac{1-u}{1+u}, \quad u=(z-1)^{\gamma} \qquad (\gamma=1/\beta),$$

so that

$$1-|w|\geqq\frac{2\,\mathrm{Re}\,u}{1+2|u|+|u|^2}.$$

Therefore we find that

$$1-|h^{-1}(1+re^{it})|\geqq\frac{\cos\gamma t}{2r^{\gamma}} \tag{3.2}$$

for real values of r and t with $r\geqq1$ and $-s<t<s$.

3.1 Our task now is to prove the following

Lemma 7. *Let $f(z)$ be an entire function of finite lower order ρ. Assume that $f(z)$ fails to take two finite values in the regions $|\arg(z-1)|<s$ and $|\pi-\arg(z+1)|<s$, where s is a positive real number with $0<s<\pi/2$. Then either $\rho\geqq\pi/(2\pi-4s)$ or else $\rho\leqq\pi/2s$.*

Proof. We may assume without loss of generality that the function $f(z)$ omits the values 0 and 1 in the regions $|\arg(z-1)|<s$ and $|\pi-\arg(z+1)|<s$. Then the composite function $F(w)=f(h(w))$, where $h(w)$ is the transformation defined with (3.1), is regular in the unit disk and fails to take the values 0 and 1 there. Hence Bohr and

Landau's theorem tells us

$$\text{(3.3)} \qquad \log|F(w)| \leqq \frac{A}{1-|w|}$$

in the unit disk, where A is a positive constant depending upon $|F(0)|$ only. It therefore follows from (3.2) and (3.3) that

$$\text{(3.4)} \qquad \log|f(1+re^{it})| \leqq 2A\frac{r^{\gamma}}{\cos\gamma t}$$

for real values of r and t with $r \geqq 1$ and $-s<t<s$. Similarly we can see that

$$\text{(3.5)} \qquad \log|f(-1-re^{it})| \leqq 2A\frac{r^{\gamma}}{\cos\gamma t}$$

for $r \geqq 1$ and $-s<t<s$.

Now for each positive r, we define the set of arguments

$$\text{(3.6)} \qquad E(r)=\{t:\log|f(re^{it})| \geqq T(r,f)/\log r\},$$

where t is understood to vary between $-\pi/2$ and $3\pi/2$. Then the spread relation [1] implies

$$\text{(3.7)} \qquad \limsup_{r\to\infty} \operatorname{meas}(E(r)) \geqq \min(\pi/\rho, 2\pi).$$

Hereafter we may further assume that

$$\text{(3.8)} \qquad \frac{1}{2}<\rho<\frac{\pi}{2\pi-4s}.$$

For otherwise, there is nothing to prove. Let s^* be a real positive number such that

$$\text{(3.9)} \qquad 2\pi-4s<s^*<\frac{\pi}{\rho}.$$

Then by virtue of (3.7), there exists a positive sequence $\{r_n^*\}$ satisfying

$$r_n^* \longrightarrow +\infty \qquad (n \to +\infty),$$

$$\mathrm{meas}(E(r_n^*)) > s^* \qquad (n \geqq 1).$$

Hence for each n, we can take an argument t_n^* of the set $E(r_n^*)$ such that $|t_n^*| \leqq (2\pi - s^*)/4$ or $|\pi - t_n^*| \leqq (2\pi - s^*)/4$. Putting

$$z_n^* = r_n^* \exp(it_n^*),$$

we thus have

$$\text{(3.10)} \qquad \log|f(z_n^*)| \geqq T(r_n^*, f)/\log r_n^* \qquad (n \geqq 1)$$

by the definition (3.6). On the other hand, since $(2\pi - s^*)/4 < s < \pi/2$, either

$$\mathrm{Re}\, z_n^* = r_n^* \cos t_n^* \geqq r_n^* \cos s$$

or else

$$\mathrm{Re}\, z_n^* = r_n^* \cos t_n^* \leqq -r_n^* \cos s.$$

Hereby for sufficiently large n, $\mathrm{Re}\, z_n^* > 1$ or $\mathrm{Re}\, z_n^* < -1$, so that we can set

$$z_n^* = 1 + r_n \exp(it_n)$$

if $\mathrm{Re}\, z_n^* > 1$, and

$$z_n^* = -1 - r_n \exp(it_n)$$

if $\mathrm{Re}\, z_n^* < -1$, where r_n is real positive and t_n is real with $-\pi/2 < t_n < \pi/2$. Evidently, $r_n^* - 1 \leqq r_n \leqq r_n^* + 1$ for $n \geqq n^*$. Therefore it is clear that

$$\text{(3.11)} \qquad r_n^*/r_n \longrightarrow 1$$

as n approaches infinity. Furthermore by means of (3.9) and (3.11), we can find a positive number t^* such that $(2\pi-s^*)/4<t^*<s$ and $|t_n|\leqq t^*$ for sufficiently large n. It then follows from (3.4) and (3.5) that

$$(3.12)\qquad \log|f(z_n^*)|\leqq 2A\frac{r_n^\gamma}{\cos\gamma t_n}\leqq 2A\frac{r_n^\gamma}{\cos\gamma t^*}$$

for $n\geqq n^*$. Here let us remark that $\gamma=1/\beta=\pi/2s$. Combining this (3.12) with (3.10), we finally obtain that

$$\frac{T(r_n^*,f)}{\log r_n^*}\leqq 2A\frac{r_n^\gamma}{\cos\gamma t^*}$$

for sufficiently large n, so that

$$\limsup_{n\to\infty}\frac{\log T(r_n^*,f)}{\log r_n^*}\leqq\gamma$$

by virtue of (3.11). Consequently the lower order ρ is less or equal to $\pi/2s$ provided it satisfies (3.8). Lemma 7 is thus proved.

3.2 We continue our discussion. Let $f(z)$ be an entire function of finite lower order ρ, and let s be a positive number with $\pi/4<s<\pi/2$. Again, suppose that the function $f(z)$ fails to take the values 0 and 1 in the regions $|\arg(z-1)|<s$ and $|\pi-\arg(z+1)||<s$. Then the above Lemma 7 implies either $\rho\geqq\pi/(2\pi-4s)$ or else $\rho\leqq\pi/2s$.

Hereafter we consider the case where $\rho\leqq\pi/2s$ holds. With a_n, we denote the zeros of $f(z)$. Then by the assumption, neither $|\arg(a_n-1)|<s$ nor $|\pi-\arg(a_n+1)|<s$. Recall that $\pi/4<s<\pi/2$. Hence it is possible to find an angular region $|\pi-\arg z|\leqq\alpha<\pi/2$ such that all but a finite number of the a_n^2 lie in this angular region. Now let us consider the auxiliary funciton defined with

$$(3.13)\qquad F(z)=f(z^{1/2})f(-z^{1/2}).$$

Then $F(z)$ is single-valued and entire, and it vanishes only at the points a_n^2. Hence it follows that

$$N(r^2, 0, F) = 2N(r, 0, f) \tag{3.14}$$

for positive values of r. Since all but a finite number of the a_n^2 satisfy $|\pi - \arg a_n^2| \leqq \alpha$, it is also possible to find a polynomial $P(z)$ such that the function $g(z) = F(-z)/P(z)$ is entire with $g(0) = 1$ and all the zeros of which satisfy $|\arg z| \leqq \alpha$. For this function $g(z)$, by means of (2.2) and (2.3), we can see that

$$\begin{aligned} 4T(r, g) &\geqq Cr + 2(1+\cos\alpha)N(r, 0, g) + S(r)\cos\alpha \\ &\geqq Cr + (\cos\alpha) r \int_0^r \frac{N(t, 0, g)}{t^2} dt \end{aligned} \tag{3.15}$$

for real positive values of r, where $C = \mathrm{Re}(g'(0))$. Here we further assume that

$$\sum_n |a_n|^{-2} = +\infty. \tag{3.16}$$

Then by (3.14), the integral

$$4\int_0^{+\infty} \frac{N(t, 0, f)}{t^3} dt = \int_0^{+\infty} \frac{N(t, 0, F)}{t^2} dt$$

diverges. Since $N(r, 0, F) - N(r, 0, g) = O(\log r)$, the integral

$$\int_0^{+\infty} \frac{N(t, 0, g)}{t^2} dt$$

also diverges. It thus follows from (3.15) that

$$\lim_{r\to\infty} \frac{T(r, g)}{r} = +\infty,$$

so that

$$\lim_{r\to\infty}\frac{T(r,F)}{r}=+\infty. \tag{3.17}$$

On the other hand by the definition (3.13), it is clear that

$$T(r^2,F)\leqq 2T(r,f)$$

for all positive r. Accordingly from (3.17), we find

$$\lim_{r\to\infty}\frac{T(r,f)}{r^2}=+\infty,$$

which means that the lower order ρ is not less than two. However this is absurd by the assumption $\rho\leqq\pi/2s<2$. Consequently, (3.16) is false. Hence the function $f(z)$ can be written in the form

$$f(z)=E(z)\exp(g(z)),$$

where $E(z)$ is the canonical product formed with the a_n as zeros, and $g(z)$ is an entire function. Since the genus of $E(z)$ is at most one, it follows that $T(r,E(z))=o(r^2)$. We therefore have

$$\liminf_{r\to\infty}\frac{T(r,\exp(g(z)))}{r^2}=\liminf_{r\to\infty}\frac{T(r,f)}{r^2}. \tag{3.18}$$

Again, by the assumption $\rho\leqq\pi/2s<2$, the right hand side of (3.18) must be equal to 0. Hereby the entire function $g(z)$ reduces to a linear function. By these facts, we finally obtain

$$\begin{aligned} f(z)f(-z)&=A^*E(z)E(-z)\\ &=A^*(-1)^m z^{2m}\prod_{a_n\neq 0}\left(1-\frac{z^2}{a_n{}^2}\right), \end{aligned}$$

where m is a non-negative integer and A^* is a suitable non-zero constant. Hence for positive real values of r,

$$\log|f(r)f(-r)|=O(\log r)+\sum_{a_n\neq 0}\log\left|1-\frac{r^2}{a_n{}^2}\right|. \tag{3.19}$$

If $\operatorname{Re} a_n^2<0$, then $\operatorname{Re} a_n^{-2}<0$. Hereby

$$\left|1-\frac{r^2}{a_n^2}\right| \geqq 1-\operatorname{Re}\frac{r^2}{a_n^2}>1$$

provided $\operatorname{Re} a_n^2<0$. Moreover since

$$\left|1-\frac{r^2}{a_n^2}\right|^2=1+\frac{r^4}{|a_n|^4}-2\operatorname{Re}\frac{r^2}{a_n^2},$$

$\operatorname{Re} a_n^2<0$ implies

$$\left|1-\frac{r^2}{a_n^2}\right|>\frac{r^2}{|a_n|^2}$$

for all positive r. It therefore follows from (3.19) that

$$\begin{aligned}\log|f(r)f(-r)| &\geqq O(\log r)+2\sum_n \log^+\frac{r}{|a_n|} \\ &=O(\log r)+2N(r,0,f)\end{aligned} \tag{3.20}$$

for positive real values of r. Here let us recall the inequalities (3.4) and (3.5). Then for each real r with $r>1$,

$$\log|f(r)f(-r)| \leqq 4A(r-1)^\gamma \tag{3.21}$$

with $\gamma=\pi/2s$. Combining (3.20) and (3.21), we at once obtain

$$N(r,0,f)=O(r^\gamma). \tag{3.22}$$

Similarly, applying the above argument to $1-f(z)$, we can also find

$$N(r,1,f)=O(r^\gamma). \tag{3.23}$$

Hence by (3.22) and (3.23), the second main theorem of Nevanlinna theory yields that

$$T(r,f)=O(r^\gamma),$$

so that the order of $f(z)$ does not exceed $\gamma=\pi/2s$.

We have therefore proved the following fact which is an improvement of the above Lemma 7.

Lemma 8. *Let $f(z)$ be an entire function, and let s be a real positive number with $\pi/4<s<\pi/2$. Suppose that $f(z)$ omits two finite values in the regions $|\arg(z-1)|<s$ and $|\pi-\arg(z+1)|<s$. Then either $\rho\geqq\pi/(2\pi-4s)$ or else $\lambda\leqq\pi/2s$, where ρ and λ indicate the lower order and the order of $f(z)$, respectively.*

3.3 Now we are in a position to piece together the foregoing results and prove the following final theorem.

Theorem 3. *Let $f(z)$ be an entire function of finite lower order. Suppose that there exist an unbounded sequence $\{w_n\}$ of complex numbers and a sequence $\{u_n\}$ of complex numbers of modulus one such that for each natural number n, all the roots of the equation $f(z)=w_n$ lie in the closed strip*

$$S_n=\{z:|\mathrm{Re}(u_n z)|\leqq 1\}.$$

Then the function $f(z)$ must be a polynomial of degree not greater than two.

Proof. Assume first that no two of the strips S_n run parallel with each other. Then for an arbitrary real number ε with $0<\varepsilon<\min(\pi/2\rho,\pi/2)$, where ρ indicates the lower order of $f(z)$, it is possible to find two natural numbers p and q such that $w_p\neq w_q$ and

$$|\arg u_p-\arg u_q|<\varepsilon. \tag{3.24}$$

With these numbers u_p and u_q, let us set

$$c=R\exp(-i(\arg u_p+\arg u_q)/2), \tag{3.25}$$

where R is a positive number with $R\cos\varepsilon/2>1$. Then from (3.24),

$$-\pi/2<\arg(u_p ce^{it})=t+(\arg u_p-\arg u_q)/2<\pi/2,$$

$$-\pi/2<\arg(u_q ce^{it})=t+(\arg u_q-\arg u_p)/2<\pi/2$$

for real values of t with $|t|<(\pi-\varepsilon)/2$. Hence it is clear that

$$\text{(3.26)}\quad \begin{aligned} \mathrm{Re}(u_p cz)&=\mathrm{Re}(u_p c)+\mathrm{Re}(u_p c(z-1))\geqq \mathrm{Re}(u_p c)>1,\\ \mathrm{Re}(u_q cz)&=\mathrm{Re}(u_q c)+\mathrm{Re}(u_q c(z-1))\geqq \mathrm{Re}(u_q c)>1 \end{aligned}$$

for values of z satisfying $|\arg(z-1)|<(\pi-\varepsilon)/2$. Similarly, if z satisfies $|\pi-\arg(z+1)|<(\pi-\varepsilon)/2$, then

$$\text{(3.27)}\quad \begin{aligned} \mathrm{Re}(u_p cz)&=-\mathrm{Re}(u_p c)+\mathrm{Re}(u_p c(z+1))\leqq -\mathrm{Re}(u_p c)<-1,\\ \mathrm{Re}(u_q cz)&=-\mathrm{Re}(u_q c)+\mathrm{Re}(u_q c(z+1))\leqq -\mathrm{Re}(u_q c)<-1. \end{aligned}$$

These facts (3.26) and (3.27) mean that the entire function $f(cz)$, where c is the constant given by (3.25), fails to take the two finite values w_p and w_q in the regions $|\arg(z-1)|<(\pi-\varepsilon)/2$ and $|\pi-\arg(z+1)|<(\pi-\varepsilon)/2$. It therefore follows from Lemma 8 that either the lower order of $f(cz)$ is not less than $\pi/2\varepsilon$ or else the order of $f(cz)$ is not greater than $\pi/(\pi-\varepsilon)$. However, since the lower order of $f(cz)$ coincides with that of $f(z)$ and $\varepsilon<\pi/2\rho$, the former does not hold. Consequently, the order of the function $f(z)$ does not exceed $\pi/(\pi-\varepsilon)$. Furthermore since ε can be chosen as small as we please, we finally obtain that the order of $f(z)$ is at most one. Now let us note Theorem 2. Then the function $f(z)$ reduces to a polynomial, because $f(z)$ satisfies the conditions of Theorem 2. Since the sequence $\{w_n\}$ is unbounded, it is clear that the degree of the polynomial $f(z)$ is at most two. This is the desired result.

It remains to consider the case when infinity many of the strips S_n coincide with each other. However, in this case our desired result follows immediately from Corollary stated in the section 1. The proof of Theorem 3 is now complete.

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