On uniquely factorizable entire functions

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This paper is a continuation of our previous paper[3]. In that paper we considered the factorization of certain entire functions and proved the following fact.

Let $A(z)$ *be* $\exp(-z)+\exp(2z)$ *and let* $B(z)$ *be* ze^z. *Then the composite function* $A(B(z))$ *is uniquely factorizable relative to the family of entire functions.*

We shall give a further investigation and present more composite entire functions which are uniquely factorizable in entire sense but not uniquely factorizable relative to the family of meromorphic functions. Our results are the following theorems.

Theorem 1. *Let* $A(z)$ *be* $\exp(-mz)+\exp(nz)$, *where* m *and* n *are distinct natural numbers, and let* $B(z)$ *be* ze^z. *Then the composite function* $A(B(z))$ *is uniquely factorizable in entire sense.*

Theorem 2. *Let* $A(z)$ *be*

$$\sum_{k=1}^{n}\{\exp(-kz)-\exp(kz)\}/k,$$

where $n>1$, *and let* $B(z)$ *be the function* ze^z. *Then the composite function* $A(B(z))$ *is uniquely factorizable relative to the family of entire functions.*

1. Let $A(z)$ be a periodic entire function of the form

$$A(z) = \sum_{k=p}^{q} a_k \exp(kz) \quad (p < 0 < q,\ a_p a_q \neq 0).$$

We assume that all the roots of equation $A(z) = A(-1/e)$ are simple, $A(0) \neq A(-1/e)$, and that $A'(x) \neq 0$ for real values of x with $x \leq -1/e$. We set $B(z) = ze^z$. Then we have the following fact.

Lemma 1. *Let $f(z)$ and $g(z)$ be nonlinear entire functions satisfying $A(B(z)) = f(g(z))$. Then $f'(g(-1)) \neq 0$.*

Proof. The four functions satisfy

$$A'(B(z))B'(z) = f'(g(z))g'(z).$$

We set the rational function $R(z) = \sum_{k=p}^{q} a_k z^k$. Then $A(z) = R(e^z)$ and

$$A'(B(z))B'(z) = R'(\exp(B(z)))\,(1+z)\exp(z+B(z)).$$

Assume that $f'(g(-1)) = 0$. Assume further that $g(w) = g(-1)$ for some point w except -1. Then $f'(g(w)) = 0$, so $A'(B(w))B'(w) = 0$. It therefore follows that $R'(\exp(B(w))) = 0$, so that $A'(B(w)) = 0$. On the other hand we have $f(g(w)) = f(g(-1))$, and hence $A(B(w)) = A(B(-1))$. This means that the equation $A(z) = A(-1/e)$ has a multiple root at $B(w)$. This contradicts the assumption on $A(z)$. Thereby $g(z)$ takes the value $g(-1)$ only at $z = -1$. If $g'(-1) = 0$, then $A'(-1/e) = 0$. This contradicts the assumption again. Consequently $g(z)$ is transcendental and has the value $g(-1)$ as an asymptotic value. Therefore $A(B(z))$ also has the value $A(B(-1))$ as an asymptotic value. Since $R(z) = a_p z^p + \cdots + a_q z^q$, $p < 0 < q$ and $A(z) = R(e^z)$, we have

$$A(B(z)) - A(B(-1)) = a_q \exp(pB(z)) \prod_{k=1}^{q-p} \{\exp(B(z)) - w_k\},$$

where $\{w_k\}$ are all the roots of equation $R(z) = A(-1/e)$. Since $A(B(z))$

converges to $A(B(-1))$ when z tends to infinity along some asymptotic curve, $\exp(B(z))$ must approach one of the values $\{w_k\}$ as z tends to infinity along this asymptotic curve. Hereby one of the values $\{\log w_k\}$ is an asymptotic value of the function $B(z)$. Since $B(z)$ has no finite asymptotic values other than zero, one of the values $\{w_k\}$ must be one, so that $R(1) = A(-1/e)$. Hence $A(0) = A(-1/e)$. This contradicts the assumption on $A(z)$. This completes the proof.

In [3] we defined the real valued function $s(x)$ for real values of x with $x \geq -1$. This function $s(x)$ is strictly increasing and continuous, and satisfies $0 \leq s(x) < \pi$, $s(-1) = 0$, $s(0) = \pi/2$, $\tan\ s(x) = -s(x)/x$ $(x \neq 0)$. Using this function $s(x)$ we defined the simply connected region

$$G = \{x + iy\ :\ x > -1,\ -s(x) < y < s(x)\}.$$

The boundary of G is the simple curve

$$z_+(x) = x + is(x),\ \ z_-(x) = x - is(x)$$

for $x \geq -1$. Evidently

$$B(x + iy) = (x + iy)\exp(x + iy) = |x + iy|\exp\{x + i(y + \arctan\ y/x)\}.$$

Thereby for any positive x, the argument of $B(x + iy)$ increases steadily from 0 to π as y varies from 0 to $s(x)$. Furthermore $B(z)$ converges uniformly to infinity as z tends to infinity from the inside of this region G. On the boundary

$$B(z_+(x)) = B(z_-(x)) = -|x + is(x)|e^x$$

for $x \geq -1$. This boundary value is real negative, and decreases strictly from $B(-1)$ to negatively infinite as x grows from -1 to positively infinite. Consequently the function $B(z)$ maps the region G conformally onto the region S whose slit is the half line $x \leq -1/e$ of the real axis. We denote the inverse function of $B(z)$ with $C(z)$.

Lemma 2. *Let $f(z)$ and $g(z)$ be nonlinear entire functions satisfying $A(B(z)) = f(g(z))$. Then $g(C(z))$ is an entire function.*

Proof. The function $g(C(z))$ is regular on the slit region S, and continuous on the two edges of the slit. It follows that

$$f(g(z_+(x))) = f(g(z_-(x)))$$

for $x \geq -1$, and $z_+(-1) = z_-(-1) = -1$. Since $f'(g(-1)) \neq 0$ by Lemma 1, $f(z)$ is univalent in a neighborhood of $g(-1)$. Hence $g(z_+(x)) = g(z_-(x))$ for $-1 \leq x < -1 + \delta$, where δ is a positive number.

Assume that $g(z_+(t)) = g(z_-(t))$ for some t with $t > -1$. Since $t > -1$, $z_+(t) \neq -1$, and hence $B'(z_+(t)) \neq 0$. Furthermore $B(z_+(t)) < B(-1) = -1/e$, so that $A'(B(z_+(t))) \neq 0$ by the assumption on $A(z)$. Thereby $f'(g(z_+(t))) \neq 0$, so that $f(z)$ is also univalent at the point $g(z_+(t))$. Hence it is possible to take a positive number δ such that $g(z_+(x)) = g(z_-(x))$ for values of x with $|x-t| < \delta$. Accordingly $g(z_+(x)) = g(z_-(x))$ for $x \geq -1$. This means that $g(C(z))$ becomes single valued on the slit. Hence from Painleve's theorem $g(C(z))$ is regular at each point of this slit. This completes the proof.

Let $f(z)$ and $g(z)$ be nonlinear entire functions which satisfy $A(B(z)) = f(g(z))$. Then $A(z) = f(g(C(z)))$. Since $g(C(z))$ is entire, this is a factorization of $A(z)$. Hence if $A(z)$ is prime in entire sense, $g(C(z))$ must be linear. We now obtain the next result.

Proposition. *Let $A(z)$ be a periodic entire function of the form*

$$A(z) = \sum_{k=p}^{q} a_k \exp(kz) \quad (p < 0 < q,\ a_p a_q \neq 0),$$

and let $B(z)$ be ze^z. Assume that all the roots of equation $A(z)=A(-1/e)$ are simple, $A(0)\neq A(-1/e)$, and $A'(x)\neq 0$ for real values of x with $x\leq -1/e$. Assume further that $A(z)$ is prime in entire sense. Then the composite function $A(B(z))$ is uniquely factorizable relative to the family of entire functions.

2. As an immediate consequence of Proposition we can prove Theorem 1. Let $A(z)$ be $\exp(-mz)+\exp(nz)$, where m and n are two distinct natural numbers. We set $R(z)=z^{-m}+z^n$. Then $A(z)=R(e^z)$, so that $A'(z)=R'(e^z)e^z$. Since $R'(z)=-mz^{-m-1}+nz^{n-1}$, all the zeros of $A'(z)$ satisfy $\exp\{(m+n)z\}=m/n$. Hence $|A(z)|=(m+n)m^{\alpha}n^{\beta}$ at every zero of $A'(z)$, where $\alpha=-m/(m+n)$ and $\beta=-n/(m+n)$. Since $R'(x)=(nx^{m+n}-m)x^{-m-1}$, $A'(x)<0$ for $x<t$ and $A'(x)>0$ for $x>t$, where $t=(m+n)^{-1}\log(m/n)$. We can easily verify that $m+x>e\log(x/m)$ for $m\geq 1$ and $x>0$. Therefore $m+n>-e\log(m/n)$, so that $-1/e<t$. Consequently $A'(x)\neq 0$ for all real values of x with $x\leq -1/e$, and $A(-1/e)>A(t)$. It is clear that $A(t)=(m+n)m^{\alpha}n^{\beta}$. Hence every root of $A(z)=A(-1/e)$ satisfies $A'(z)\neq 0$. Furthermore $x\geq e\log x$ for $x>0$. Hereby $2\geq e\log 2$, so that $\log A(-1/e)>m/e\geq\log 2$ for the case $m\geq 2$. Hence $A(-1/e)>A(0)$ in this case. Since $\log(1+x)>x-x^2/2$ for $x>0$ and $e>5/2$, $\log\{(5^{1/2}+1)/2\}>1/e$. Furthermore since $x^{-2}+x<2$ for $1<x<(5^{1/2}+1)/2$, $\exp(-2/e)+\exp(1/e)<2$. Hence for the case $m=1$, $n\geq 2$ and

$$A(-1/e)=\exp(1/e)+\exp(-n/e)\leq\exp(1/e)+\exp(-2/e)<A(0).$$

The function $A(z)=\exp(-mz)+\exp(nz)$ is surely prime in entire sense [2]. Consequently the function $A(z)$ satisfies the hypotheses of Proposition. Thus $A(B(z))$ is uniquely factorizable in entire sense. This completes the proof of Theorem 1.

3. We next prove Theorem 2. Let $A(z)$ be the periodic function

$$\sum_{k=1}^{n} \{\exp(-kz) - \exp(kz)\}/k$$

with $n > 1$. For convenience we set $R(z) = \sum_{k=1}^{n} (z^{-k} - z^{k})/k$. Then $A(z) = R(e^{z})$ and

$$R'(z) = \sum_{k=1}^{n} (-z^{-k-1} - z^{k-1}) = (z^{n+1}+1)(z^{n}-1)(1-z)^{-1}z^{-n-1}.$$

All the zeros of $R'(z)$ satisfy $z^{n} = 1$ or $z^{n+1} = -1$. Conversely the points satisfying $z^{n+1} = -1$ or $z^{n} = 1$ except $z = 1$ fulfill $R'(z) = 0$. For the case n is even, the point $z = -1$ is a zero of $R'(z)$ and its multiplicity is two. For the case n is odd, all the zeros of $R'(z)$ are simple and $R'(-1) \neq 0$. Clearly $R'(x) < 0$ for real values of x with $x > 0$. Since $A'(z) = R'(e^{z})e^{z}$, $A'(x) < 0$ for real x. Especially $A(0) \neq A(-1/e)$. The function $R(z)$ takes pure imaginary values on $|z| = 1$. Hence $R(z)$ is pure imaginary at each point z satisfying $R'(z) = 0$. Since $R(\exp(-1/e))$ is real and not zero, all the roots of $R(e^{z}) = R(\exp(-1/e))$ are simple. Thereby every point which satisfies $A(z) = A(-1/e)$ is not a zero of $A'(z)$.

In order to complete the proof we need the following

Theorem 3. *The periodic entire function*

$$\sum_{k=1}^{n} \{\exp(-kz) - \exp(kz)\}/k,$$

where $n > 1$, is prime relative to the family of entire functions.

The function $A(z)$ also satisfies the hypotheses of Proposition, so that the composite function $A(B(z))$ is uniquely factorizable in entire sense.

4. Our task is to show the above Theorem 3. Let $f(z)$ and $g(z)$ be entire functions which satisfy $A(z) = f(g(z))$. Assume that $f(z)$ is transcendental. Let w be a zero of $f'(z)$. Then all the roots of $g(z) = w$ satisfy $A'(z) = 0$. Hence they fulfill $|e^z| = 1$. Furthermore all the points which satisfy $\exp\{(n+1)z\} = -1$ or $\exp(nz) = 1$ except $e^z=1$ are zeros of $f'(g(z))g'(z)$. Hence if $f'(z)$ has only finitely many zeros, then the orders of $f(z)$ and $g(z)$ are both not less than one. Thus $f(g(z))$ has infinite order [4]. This is absurd. Accordingly $f'(z)$ has infinitely many zeros. It therefore follows from Edrei's theorem [1] that the right factor $g(z)$ is a polynomial degree at most two. If its degree is two, we may assume that $g(z) = z(z+s)$. Since $g(z) = g(-z-s)$ and $A(z) = -A(-z)$, $A(z) = -A(z+s)$. Accordingly $R(e^z) = -R(e^{z+s})$, so that $\exp(ks) = -1$ for $k = 1,\dots,n$. Since $n > 1$, $e^s = e^{2s} = -1$. This is absurd again. Consequently if $f(z)$ is transcendental, $g(z)$ is linear.

Hereafter we assume that $f(z)$ is a polynomial of degree at least two. Since $f(g(z))$ is of exponential type and periodic with period $2\pi i$, the right factor $g(z)$ is also exponential type and periodic with period $2\pi\nu i$, where ν is a positive integer. Hence $g(z)$ can be written as

$$g(z) = \sum_{k=p}^{q} c_k \exp(kz/\nu).$$

Since $A(z) = R(e^z)$ and $A(z) = f(g(z))$, we consequently have a factorization

$$R(z^\nu) = P(Q(z)), \quad P(z) = \sum_{k=0}^{m} \alpha_k z^k, \quad Q(z) = \sum_{k=p}^{q} r_k z^k,$$

where ν is a positive integer and the degree m is not less than two.

It follows by definition that $\nu n = mq$ and $-\nu n = mp$. Since $m \geq 2$, $P'(z)$ has a zero. Hence for some finite value w, all the roots of equation $Q(z) = w$ are distributed on $|z| = 1$ only. Therefore $\bar{Q}(1/\bar{z}) - \bar{w} = \varepsilon\{Q(z) - w\}$ with $|\varepsilon| = 1$. Thus we may assume without loss of generality that

$$Q(z) = \sum_{k=-q}^{q} r_k z^k \quad (r_q r_{-q} \neq 0,\ q \geq 1)$$

and the coefficients r_k satisfy $r_k = \bar{r}_{-k}$, and $r_0 = 0$. Evidently on $|z| = 1$, $Q(z)$ takes only real values, while $R(z^\nu)$ takes only pure imaginary values. Hence the polynomial $P(z)$ takes only pure imaginary values on the real axis, so that all the coefficients α_k are pure imaginary.

5. We consider the case $\nu = 1$. In this case $R(z) = P(Q(z))$ and $n = mq$. Evidently $\alpha_m(r_q)^m = -1/n$ and $\alpha_m(r_{-q})^m = 1/n$, so that $(r_q/r_{-q})^m = -1$. Assume for a moment that $q = 1$. Since $r_0 = 0$, the $(n-1)$th term of $P(Q(z))$ is $\alpha_{m-1}(r_q)^{m-1}$, and the $(1-n)$th term is $\alpha_{m-1}(r_{-q})^{m-1}$. Hence $\alpha_{m-1}(r_q)^{m-1} = 1/(1-n)$ and $\alpha_{m-1}(r_{-q})^{m-1} = 1/(n-1)$. Thus $(r_q/r_{-q})^{m-1} = -1$. This is absurd. We therefore assume that $q > 1$. By $R(z) = P(Q(z))$, for each k with $(m-1)q < k < mq$,

$$\alpha_m\{\sum_{j_1 + j_2 + \cdots + j_m = k} r_{j_1} r_{j_2} \cdots r_{j_m}\} = -1/k.$$

We set $k = mq-1$. Then $\alpha_m m(r_q)^{m-1} r_{q-1} = 1/(1-n)$, and hence $r_{q-1}/r_q = q/(n-1)$. Since $r_j = \bar{r}_{-j}$, $r_{1-q}/r_{-q} = q/(n-1)$. Furthermore by induction, for each k with $1 \leq k < q$, the ratio r_k/r_q is real and $r_{-k}/r_{-q} = r_k/r_q$.

The $(m-1)q$th term of $P(Q(z))$ is

$$\alpha_{m-1}(r_q)^{m-1} + \alpha_m\{\sum_{j_1 + j_2 + \cdots + j_m = n-q} r_{j_1} r_{j_2} \cdots r_{j_m}\}$$

and is equal to $1/(q-n)$. Since each subscript j_i runs from $-q$ to q, if $j_1+j_2+\cdots+j_m=(m-1)q$, then $j_1\geq 0$, $j_2\geq 0$, ..., $j_m\geq 0$. Since r_k/r_q is real for $1\leq k<q$ and $r_0=0$, it thus follows that

$$\{\Sigma\ r_{j_1}r_{j_2}\cdots r_{j_m}\}=C(r_q)^m$$
$$j_1+j_2+\cdots+j_m=n-q$$

with some real number C, so that $\alpha_{m-1}(r_q)^{m-1}$ is real. Since each α_j is pure imaginary and $\alpha_m(r_q)^m$ is nonzero real, we thus have $\alpha_{m-1}=0$. It therefore follows that for each k with $(m-2)q<k\leq(m-1)q$

$$\alpha_m\{\Sigma\ r_{j_1}r_{j_2}\cdots r_{j_m}\}=-1/k,$$
$$j_1+j_2+\cdots+j_m=k$$

and hence, by induction again, every ratio r_j/r_q is real for $-q<j\leq 0$. Consequently every ratio r_j/r_q is real for $-q<j\leq q$, so that the ratio r_{-q}/r_q is real. Since $(r_q/r_{-q})^m=-1$, $r_q/r_{-q}=-1$ and the degree m must be odd. Further since $r_{-q}=\overline{r}_q$, $r_q=-r_{-q}=-\overline{r}_q$, that is, r_q is pure imaginary. Thereby every coefficient r_j is pure imaginary and $r_j=\overline{r}_{-j}=-r_{-j}$. It thus follows that $Q(1)=0$ and $Q(-1)=0$. Since $R'(1)=P'(Q(1))Q'(1)\neq 0$, we have $P'(0)\neq 0$. If n is even, $R'(-1)=R''(-1)=0$. Hence $Q'(-1)=Q''(-1)=0$. The derivative $Q'(z)$ has at most $2q-1$ distinct zeros. We denote them with $s_j=\exp(i\zeta_j)$ $(j=1,2,\ldots,s)$ where $1\leq s\leq 2q-1$ and $0<\zeta_1<\zeta_2<\cdots<\zeta_s<2\pi$. Evidently $Q(e^{i\theta})$ takes real values only and varies monotonously on each interval $\zeta_j<\theta<\zeta_{j+1}$ $(j=1,2,\ldots,s,\ \zeta_{s+1}=2\pi+\zeta_1)$. Hence $P'(Q(e^{i\theta}))$ has at most $m-1$ distinct zeros there. Consequently $R'(z)$ has at most $(m-1)s+s$ distinct zeros on $|z|=1$. Note that $ms\leq(2q-1)m=2n-m<2n-1$, and that $R'(z)$ has exactly $2n-1$ distinct zeros on $|z|=1$ when n is even. This is impossible. Hence the degree n is odd, so that the degree q is also odd.

Since n is odd, $R'(-1) \neq 0$ and all the zeros of $R'(z)$ are simple. We can write the zeros of $R'(z)$ explicitly. We set $z_k = \exp\{ik\pi/(n+1)\}$ for odd k with $1 \leq k \leq n$, $z_k = \exp(ik\pi/n)$ for even k with $1 < k < n$, and $z_k = \bar{z}_{2n-k+1}$ for $n+1 \leq k \leq 2n$. Then these $2n$ points are the zeros of $R'(z)$. Clearly the arguments of z_k increase with k for $1 \leq k \leq n$. Evidently

$$Q(e^{i\theta}) = \sum_{k=1}^{q} 2ir_k \sin\ k\theta,$$

so that $Q(z_j) = -Q(z_{2n-j+1})$ for $1 \leq j \leq 2n$.

The derivative $Q'(z)$ has exactly $2q$ zeros. As before, we denote them with $s_j = \exp(i\zeta_j)$ $(1 \leq j \leq 2q)$ such that $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{2q} < 2\pi$. Since

$$Q'(e^{i\theta})e^{i\theta} = \sum_{k=1}^{q} 2kr_k \cos\ k\theta,$$

we have $\zeta_j + \zeta_{2q-j+1} = 2\pi$ for $1 \leq j \leq 2q$. Let w_j $(j = 1,2,\ldots,m-1)$ be the zeros of $P'(z)$. On each interval $\zeta_j < \theta < \zeta_{j+1}$$(j = 1,2,\ldots,2q,\ \zeta_{2q+1} = 2\pi + \zeta_1)$, $Q(e^{i\theta})$ takes these $m-1$ values w_j at most once. However since $2(m-1)q + 2q = 2n$, $Q(e^{i\theta})$ must take each w_j exactly once there. If $s_1 = z_h$, then $s_{2q} = \bar{s}_1 = z_{2n-h+1}$. Hence the minor arc of $|z| = 1$ between s_{2q} and s_1 contains $2(h-1)$ points of $\{z_j\}$. Hereby $2h = m+1$. Thus we can set $s_j = z_{jm+l}$ for $j = 1,2\ldots,2q$, where $l = (1-m)/2$.

We now calculate the sum $s_1 + s_2 + \cdots + s_{2q}$. Since $\bar{s}_j = s_{2q-j+1}$ for $1 \leq j \leq 2q$,

$$\sum_{j=1}^{2q} s_j = \sum_{j=1}^{q} (s_j + \bar{s}_j) = 2\sum_{j=1}^{q} \cos\ \zeta_j.$$

Since $(jm+l) + \{(q-j+1)m + l\} = n+1$, if $jm+l$ is odd, $(q-j+1)m+l$ is also odd, and hence $\zeta_j + \zeta_{q-j+1} = \pi$ by definition. Accordingly if l

is odd, then

$$\sum_{j=1}^{2q} s_j = 2\sum_{j=1}^{(q+1)/2} \cos\ \zeta_{2j-1} = 2\sum_{j=1}^{(q+1)/2} \cos(2jm - m + l)\pi/n$$
$$= -\sin(\pi/2n)/\sin(\pi/2q),$$

and if l is even,

$$\sum_{j=1}^{2q} s_j = 2\sum_{j=1}^{(q-1)/2} \cos\ \zeta_{2j} = 2\sum_{j=1}^{(q-1)/2} \cos(2jm + l)\pi/n$$
$$= -\sin(\pi/2n)/\sin(\pi/2q).$$

Consequently we obtain

$$\sum_{j=1}^{2q} s_j = -\sin(\pi/2n)/\sin(\pi/2q).$$

On the other hand it follows from definition that

$$Q'(z) = \sum_{k=-q}^{q} kr_k z^{k-1} = qr_q z^{-q-1} \prod_{j=1}^{2q} (z-s_j),$$

so that

$$(q-1)r_{q-1} = -qr_q \sum_{j=1}^{2q} s_j.$$

Since $r_{q-1}/r_q = q/(n-1)$, we obtain

$$\sum_{j=1}^{2q} s_j = -(q-1)/(n-1).$$

The function $(t-1)\sin(\pi/2t)$ is strictly increasing for $t > 1$. Hence

$$(n-1)\sin(\pi/2n) > (q-1)\sin(\pi/2q).$$

This is a contradiction. The case $\nu = 1$ never occurs.

6. We next consider the case $\nu > 1$ and $q = 1$. For convenience we set $Q(z) = az + bz^{-1}$. We may assume that $ab \neq 0$ and $a = \bar{b}$. Since $R(z^{\nu}) = P(Q(z))$, $m = \nu n$ and $\alpha_m a^m = -1/n$, $\alpha_m b^m = 1/n$. Therefore $(a/b)^m = -1$. Since $Q'(z) = a - bz^{-2}$, the derivative $Q'(z)$ vanishes at $\pm e^{i\gamma}$, where $0 \leq \gamma = \arg\ b < 2\pi$. The points $\pm e^{i\gamma}$ are also zeros of $R'(z^{\nu})$.

Therefore $(\pm e^{i\gamma})^m = 1$ or $(\pm e^{i\gamma})^{\nu+m} = -1$. Since $(b/a)^m = (e^{i2\gamma})^m = -1$, $e^{i\gamma m} = \pm i$. Hence $(\pm e^{i\gamma})^{\nu+m} = -1$, so that $e^{i\gamma\nu} = e^{i\gamma m} = \pm i$. Since $m = \nu n$, $\pm i = e^{i\gamma m} = (e^{i\gamma\nu})^n = (\pm i)^n$. The degree n must be odd. The derivative $R'(z^\nu)$ has exactly $2\nu n$ zeros. They are $z_j = \exp(i2j\pi/m)$ for $1 \le j \le m$ except multiples of n, $\zeta_j = \exp\{i(2j-1)\pi/(\nu+m)\}$ for $1 \le j \le \nu + m$. Since the zero $e^{i\gamma}$ satisfies $(e^{i\gamma})^{\nu+m} = -1$, we can set $e^{i\gamma} = \zeta_k$ for some k with $1 \le k \le \nu + m$, that is, $\gamma = (2k-1)\pi/(\nu+m)$. Since γ is the argument of b and $a = \bar{b}$,

$$Q(e^{i\theta}) = 2\,|\,a\,|\cos(\theta - \gamma).$$

As before we denote the $m-1$ zeros of $P'(z)$ with $w_j (j = 1,2,\ldots,m-1)$. These are real and distinct. The function $Q(z)$ takes these zeros w_j exactly twice on $|\,z\,| = 1$. Thereby the union of the sets

$$\{Q(z_j)\colon\ 1 \le j \le m\} - \{Q(z_{jn})\colon\ 1 \le j \le \nu\},$$

$$\{Q(\zeta_j)\colon\ 1 \le j \le \nu + m\} - \{Q(\pm e^{i\gamma})\}$$

coincides with the set $\{w_j\colon\ 1 \le j \le m-1\}$. It is clear that the latter set consists of $(\nu + m - 2)/2$ values, that is, the latter set is

$$\{2\,|\,a\,|\cos(2j\pi/(\nu+m))\colon\ 1 \le j \le (\nu+m-2)/2\}.$$

If $Q(z_s) = Q(z_t)$ for some $1 \le s < t \le m$, then

$$\{(s+t)/m\} - \{(2k-1)/(\nu+m)\} = N$$

is an integer. Since $m = \nu n$, $(s+t)(n+1) = \{2k - 1 + N(\nu+m)\}n$. The degree n is odd and $(\nu+m) = \nu(n+1)$, so the right side is odd while the left side is even. This is impossible. Consequently the values $Q(z_j)$ $(j = 1,2,\ldots m)$ are all distinct. Hence the former set consists of $m - \nu$ values. We assume that $Q(z_j) = Q(\zeta_l)$ for some j and l with $1 \le j \le m$ and $1 \le l \le \nu + m$. Then

$$(2j/m) - \{(2l-1)/(\nu+m)\} \text{ or } (2j/m) + \{(2l - 4k + 1)/(\nu+m)\}$$

is an even integer. However we have a contradiction by the same reason as above. Thereby the two sets have no common values, so that the union consists of $\{(\nu+m-2)/2\}+(m-\nu)$ values. Evidently this number is greater than $m-1$. This is impossible. Consequently the case $\nu>1$ and $q=1$ never occurs.

7. We finally consider the case $\nu>1$ and $q>1$. Since $R(z^{\nu})=P(Q(z))$, $\nu n=mq$, $\alpha_m(r_q)^m=-1/n$ and $\alpha_m(r_{-q})^m=1/n$. For each k with $(m-1)q<k<mq$, the kth term of $P(Q(z))$ is

$$\alpha_m\{\sum_{j_1+j_2+\cdots+j_m=k} r_{j_1}r_{j_2}\cdots r_{j_m}\}.$$

This coefficient is zero when k is not a multiple of ν, while this is equal to $-\nu/k$ when k is a multiple of ν. We denote by S the set of integers

$$\{j: q-j \text{ is not a multiple of } \nu\}.$$

We can see that the ratio r_j/r_q is real for $1\leq j<q$, and that $r_j=0$ for $j\in S$ with $1\leq j<q$. Indeed the $(mq-1)$th term of $P(Q(z))$ is $\alpha_m mr_{q-1}(r_q)^{m-1}$ and is equal to zero. Hence r_{q-1}=0. Let l be an integer with $1\leq l<q-1$. Assume that r_j/r_q is real for $l<j<q$, and $r_j=0$ for $j\in S$ with $l<j<q$. The $(mq-q+l)$th term of $P(Q(z))$ is

$$\alpha_m\{\sum_{j_1+j_2+\cdots+j_m=mq-q+l} r_{j_1}r_{j_2}\cdots r_{j_m}\}.$$

Every subscript does not exceed q. Hence $l\leq j_i\leq q$ for $i=1,2,\ldots,m$, so that by the inductive assumption, the $(mq-q+l)$th term becomes

$$\alpha_m mr_l(r_q)^{m-1}+C\alpha_m(r_q)^m,$$

where C is a real constant. Since this term is real, the ratio r_l/r_q is real. Note that $j_1+j_2+\cdots+j_m=mq-q+l$ becomes $(q-j_1)+(q-j_2)+\cdots+(q-j_m)=q-l$. Hence if l belongs to S, at least one j_i also belongs to S. Therefore the $(mq-q+l)$th term becomes $\alpha_m m r_l (r_q)^{m-1}$, and this is zero because $mq-q+l$ is not a multiple of ν. This implies that $r_l=0$ when l belongs to S.

We consider the case $\nu \geq q$. Every j with $1 \leq j < q$ satisfies $0 < q-j < \nu$ in this case. Hence $j \in S$, so that $r_j = 0$ for $1 \leq j < q$. Since $r_j = \bar{r}_{-j}$, $Q(z) = r_q z^q + r_{-q} z^{-q}$. The νth term of $R(z^\nu)$ does not vanish, so the integer ν is a multiple of q. Accordingly $R(z^\mu) = P(r_q z + r_{-q} z^{-1})$ with $\mu = \nu/q$. This is just the case $q = 1$. Hence we can reject the case $\nu \geq q$.

We next consider the case $\nu < q$ and q is a multiple of ν. In this case if j is not a multiple of ν, j belongs to S, and hence $r_j = 0$ and $r_{-j} = 0$. Therefore we can define a rational function $Q_*(z)$ such that $Q_*(z^\nu) = Q(z)$. Hereby $R(z) = P(Q_*(z))$, which is the case $\nu = 1$. We thus reject this case.

Finally we treat the case $\nu < q$ and q is not a multiple of ν. For this case, $mq - q < mq - \nu$, so that the $(mq-\nu)$th term of $P(Q(z))$ is

$$\alpha_m \{\Sigma r_{j_1} r_{j_2} \cdots r_{j_m}\}, \quad j_1 + j_2 + \cdots + j_m = mq - \nu$$

and is equal to $-1/(n-1)$. Since $r_j = 0$ for $j \in S$ with $1 \leq j < q$, this term becomes $\alpha_m m r_{q-\nu} (r_q)^{m-1}$. It thus follows that $r_{q-\nu}/r_q = r_{\nu-q}/r_{-q} = n/(mn-m)$. The integer $mq - q$ is not a multiple of ν. Hence the $(mq-q)$th term of $P(Q(z))$ vanishes, so that

$$\alpha_{m-1}(r_q)^{m-1} + \alpha_m \{\Sigma r_{j_1} r_{j_2} \cdots r_{j_m}\} = 0. \quad j_1 + j_2 + \cdots + j_m = mq - q$$

Furthermore the second term of left side vanishes. Thereby $\alpha_{m-1} = 0$, and hence for each k with $(m-2)q < k < (m-1)q$, the kth term of $P(Q(z))$ is

$$\alpha_m\{\Sigma r_{j_1} r_{j_2} \cdots r_{j_m}\}.$$
$$j_1 + j_2 + \cdots + j_m = k$$

Since $R(z^{\nu}) = P(Q(z))$, this quantity is zero when k is not a multiple of ν, while this is equal to $-\nu/k$ if k is a multiple of ν. Therefore by the same reason as above, r_j/r_q is real for $-q < j \leq -1$ and $r_j = 0$ for $j \in S$ with $-q < j \leq -1$. Since $r_{\nu-q}/r_{-q} = n/(mn-m)$, $r_{-q}/r_q = (r_{-q}/r_{\nu-q})(r_{\nu-q}/r_q)$ is real. Since $r_{-q} = \bar{r}_q$, $r_{-q}/r_q = \pm 1$. Furthermore since $(r_{-q}/r_q)^m = -1$, the degree m must be odd and $r_{-q}/r_q = -1$. On the other hand, since $r_{\nu-q} \neq 0$, the integer $\nu - q$ does not belong to the set S. Hence $2q$ is a multiple of ν, so that $2q = k\nu$ with some integer k. Since q is not a multiple of ν, the integer k must be odd. Since $mq = \nu n$, we further have $2n = km$. This is absurd, because km is odd. Consequently the case $\nu > 1$ and $q > 1$ does not occur. The proof of Theorem 3 is now complete.

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