Remarks on the Growth of Meromorphic Functions

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Let $f(z)$ be a meromorphic function and let $T(r, f)$ be its characteristic function. Then the order λ and the lower order μ of the function $f(z)$ are defined by the relations

$$\lambda=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}, \quad \mu=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

respectively. If the function $f(z)$ has finite order, the concept of genus q can be defined [10]. The genus q is zero or a positive integer and satisfies $q\leq\lambda\leq q+1$ in general.

It is known that the growth of a meromorphic function is closely related to the distribution of its zeros and poles. For instance a simple behavior of the arguments of the zeros and poles of a meromorphic function is almost enough to bound the growth of the function (cf. [1], [3], [4], [5], [7]). In particular the properties of meromorphic functions with negative zeros and positive poles have been investigated extensively (cf. [8], [11], [12]).

The purpose of this note is to investigate the growth of meromorphic functions whose zeros and poles obey some simple geometrical restrictions on their position. We shall prove the following results.

Theorem 1. *Let $f(z)$ be a meromorphic function of genus zero. If all its zeros $\{a_\nu\}$ satisfy $|\pi-\arg a_\nu|\leq\gamma$ and all its poles $\{b_\nu\}$ satisfy $|\arg b_\nu|\leq\gamma$, where γ is some real number with $0\leq\gamma<\pi/2$, then*

$$\liminf_{r\to\infty}\frac{N(r,0,f)+N(r,f)}{T(r,f)}\leq\frac{2(1-\lambda^2)}{1-\lambda^2+\lambda^2\cos\gamma},$$

where λ indicates the order of the function $f(z)$.

Theorem 2. *Let $f(z)$ be a meromorphic function of finite genus $q \geq 1$. Assume that all its zeros $\{a_\nu\}$ and poles $\{b_\nu\}$ satisfy $|\pi - \arg a_\nu| \leq \gamma$ and $|\arg b_\nu| \leq \gamma$, respectively, where γ is some real number satisfying $0 \leq 2q\gamma < \pi$. Then the lower order μ of the function $f(z)$ satisfies $\mu \geq 2[(q-1)/2]+1$, and*

$$\liminf_{r\to\infty} \frac{N(r,0,f)+N(r,f)}{T(r,f)} \leq \frac{2(\mu^2-p^2)}{\mu^2-p^2+\mu^2\cos p\gamma},$$

where p is the integer $2[(q-1)/2]+1$.

Furthermore if the real number γ satisfies $0 \leq 2(q+2)\gamma < \pi$,

$$\liminf_{r\to\infty} \frac{N(r,0,f)+N(r,f)}{T(r,f)} \leq \frac{2(s^2-\lambda^2)}{s^2-\lambda^2+\lambda^2\cos s\gamma},$$

where s is the integer $2[(q+1)/2]+1$, and λ denotes the order of $f(z)$.

J. Williamson [12] already showed the inequality $\mu \geq 2[(q-1)/2]+1$ in the class of meromorphic functions having negative zeros and positive poles. Further he mentioned without proof that for any positive even integer q and for any real λ, μ with $q-1 < \mu \leq q \leq \lambda < q+1$, there exists a meromorphic function of order λ and lower order μ whose zeros lie on the negative real axis and whose poles lie on the positive real axis. Hence the lower bound $2[(q-1)/2]+1$ is sharp. For completeness we shall present such meromorphic functions in the final section.

An immediate consequence of Theorems 1 and 2 is the following

Corollary 1. *Let $f(z)$ be a meromorphic function of finite genus q. Assume that either the order or the lower order of the function $f(z)$ is an odd integer, and that all its zeros $\{a_\nu\}$ and poles $\{b_\nu\}$ satisfy $|\pi - \arg a_\nu| \leq \gamma$ and $|\arg b_\nu| \leq \gamma$, respectively, with some real γ satisfying $0 \leq 2(q+2)\gamma < \pi$. Then*

$$\liminf_{r\to\infty} \frac{N(r,0,f)+N(r,f)}{T(r,f)} = 0.$$

In particular, the function $f(z)$ has no finite deficient values other than zero.

Let q be an arbitrary positive even integer, and let $\{a_\nu\}$ be the sequence of real positive numbers defined with $a_\nu = \nu^{1/q}$ ($\nu \geq 1$). Then the meromorphic function

$$f(z) = \prod_{\nu \geq 1} \frac{E(-z/a_\nu, q)}{E(z/a_\nu, q)}$$

is of order q, of regular growth, and satisfies

$$\lim_{r \to \infty} \frac{N(r,0,f) + N(r,f)}{T(r,f)} = \frac{2}{q+1}.$$

Hence in the above Corollary 1 we cannot replace *an odd integer* by *a positive integer*. We shall calculate these facts explicitly in the final section.

The above Theorem 2 is easily extended to entire functions. The next result, stated without proof, follows readily from the methods of this paper.

Theorem 3. *Let $f(z)$ be an entire function of finite genus $q \geq 1$. Assume that all its zeros lie in the sector $|\arg z| \leq \gamma$, where γ is some real number satisfying $0 \leq 2q\gamma < \pi$. Then the lower order μ of the function $f(z)$ satisfies $\mu \geq q$, and*

$$\liminf_{r \to \infty} \frac{N(r,0,f)}{T(r,f)} \leq \frac{2(\mu^2 - q^2)}{\mu^2 - q^2 + \mu^2 \cos q\gamma}.$$

Furthermore if the number γ satisfies $0 \leq 2(q+1)\gamma < \pi$,

$$\liminf_{r \to \infty} \frac{N(r,0,f)}{T(r,f)} \leq \frac{2\{(q+1)^2 - \lambda^2\}}{(q+1)^2 - \lambda^2 + \lambda^2 \cos(q+1)\gamma},$$

where λ indicates the order of the function $f(z)$.

It should be remarked that the inequality $\mu \geq q$ is already proved and this lower bound q is no longer true when the opening of the sector exceeds π/q (cf. [9]).

From Theorems 1 and 3 we obtain the following fact which is an improvement of a result of Shea [11, Corollary 2.2].

Corollary 2. *Let $f(z)$ be an entire function of finite genus q. Assume that either the order or the lower order of the function $f(z)$ is a positive integer, and that all its zeros lie in the sector $|\arg z| \leq \gamma$ with γ satisfying $0 \leq 2(q+1)\gamma < \pi$. Then the function $f(z)$ has no finite deficient values other zero.*

1. Lemmas. Our starting point will be the following elementary fact.

Lemma 1. *Let $f(z)$ be meromorphic with $f(0)=1$. Assume that all its zeros $\{a_\nu\}$ and poles $\{b_\nu\}$ satisfy*

$$|\pi - \arg a_\nu| \leq \gamma < \pi/2, \quad |\arg b_\nu| \leq \gamma < \pi/2$$

respectively. Then we have

$$4T(r, f) \geq 2(1+\cos\gamma)N(r) + r\cos\gamma \int_0^r (t^{-2} - r^{-2})N(t)dt + Cr$$

for all values of $r>0$, where $N(r)=N(r, 0, f)+N(r, f)$ and C is a constant.

Proof. It is well known [10, p. 222] that

$$\begin{aligned}\pi f'(0)r = &\int_0^{2\pi} \log|f(re^{it})| e^{-it}dt \\ &- \pi \sum_{|a_\nu|<r} \left(\frac{r}{a_\nu} - \frac{\bar{a}_\nu}{r}\right) + \pi \sum_{|b_\nu|<r} \left(\frac{r}{b_\nu} - \frac{\bar{b}_\nu}{r}\right)\end{aligned}$$

for $r>0$. Since $|\pi - \arg a_\nu| \leq \gamma$, it follows that

$$\begin{aligned}\sum_{|a_\nu|<r} \operatorname{Re}\left(\frac{r}{a_\nu} - \frac{\bar{a}_\nu}{r}\right) &\leq \sum_{|a_\nu|<r} \left(\frac{|a_\nu|}{r} - \frac{r}{|a_\nu|}\right)\cos\gamma \\ &= \cos\gamma \int_0^r \left(\frac{t}{r} - \frac{r}{t}\right)dn(t, 0).\end{aligned}$$

Similarly, since $|\arg b_\nu| \leq \gamma$, it also follows that

$$\begin{aligned}\sum_{|b_\nu|<r} \operatorname{Re}\left(\frac{r}{b_\nu} - \frac{\bar{b}_\nu}{r}\right) &\geq \sum_{|b_\nu|<r} \left(\frac{r}{|b_\nu|} - \frac{|b_\nu|}{r}\right)\cos\gamma \\ &= \cos\gamma \int_0^r \left(\frac{r}{t} - \frac{t}{r}\right)dn(t, \infty).\end{aligned}$$

We therefore obtain

$$C\pi r \geq \int_0^{2\pi} \log|f(re^{it})| \cos t\,dt + \pi\cos\gamma \int_0^r \left(\frac{r}{t} - \frac{t}{r}\right) dn(t),$$

where $n(t) = n(t, 0) + n(t, \infty)$ and C is the real part of $f'(0)$. By a rough estimation and Jensen's formula,

$$\left|\int_0^{2\pi} \log|f(re^{it})| \cos t\,dt\right| \leq \int_0^{2\pi} |\log|f(re^{it})||\,dt = 2\pi\{2T(r,f) - N(r)\}$$

On the other hand, by the definition of $N(r)$, we obtain

$$\int_0^r t^{-1} dn(t) = r^{-1}n(r) + r^{-1}N(r) + \int_0^r t^{-2}N(t)dt,$$

$$\int_0^r t\,dn(t) = rn(r) - rN(r) + \int_0^r N(t)dt$$

for $r>0$. Consequently

$$Cr \geq -2\{2T(r,f) - N(r)\} + \{2N(r) + \int_0^r (rt^{-2} - r^{-1})N(t)dt\}\cos\gamma$$

for all values of $r>0$, which proves Lemma 1.

By this Lemma 1 we at once have

$$4T(r,f) \geq r\cos\gamma \int_0^{r/2} (t^{-2} - r^{-2})N(t)dt + Cr$$
$$\geq C' \, r \int_0^{r/2} t^{-2}N(t)dt + Cr$$

for $r>0$, where C' is a positive constant. Hence if the integral $\int_0^{+\infty} t^{-2}N(t)dt$ is divergent, $T(r, f)/r$ tends to infinity with r. In particular the lower order μ of the function $f(z)$ satisfies $\mu \geq 1$.

Lemma 2. *Let the assumptions of Lemma* 1 *be satisfied. Assume further that the zeros* $\{a_\nu\}$ *and poles* $\{b_\nu\}$ *satisfy*

$$\sum_\nu |a_\nu|^{-1} + \sum_\nu |b_\nu|^{-1} = +\infty.$$

Then the lower order μ *of the function* $f(z)$ *is not less than one.*

Furthermore if μ *is finite,*

$$\liminf_{r\to\infty}\frac{N(r,0,f)+N(r,f)}{T(r,f)}\leq\frac{2(\mu^2-1)}{\mu^2-1+\mu^2\cos\gamma}.$$

Proof. By assumption the integral $\int_0^{+\infty} t^{-2}N(t)dt$ is divergent, where $N(r)=N(r, 0, f)+N(r, f)$. Hence we have already proved the first statement.

Now we shall prove the second statement. We assume for a moment that the order of the function $N(r)$ is less than that of $T(r, f)$. Then it is clear that

$$\liminf_{r\to\infty}\frac{N(r,0,f)+N(r,f)}{T(r,f)}=0,$$

so that the desired inequality holds.

Henceforth we may assume that the order of $N(r)$ is equal to the order of the function $T(r, f)$. Evidently the lower order of $N(r)$ does not exceed the lower order μ of $T(r, f)$. Hereby for the function $N(r)$, there exists a sequence $\{r_\nu\}$ of Polya peaks of the second kind of order μ [11, p. 208]. Let $\{r'_\nu\}$ and $\{r''_\nu\}$ be the associated sequences. Then the three sequences $\{r'_\nu\}$, $\{r_\nu/r'_\nu\}$, $\{r''_\nu/r_\nu\}$ converge to infinity as ν tends to infinity, and

$$N(t)\geq(1+o(1))(t/r_\nu)^\mu N(r_\nu)$$

for $r'_\nu\leq t\leq r''_\nu$. It therefore follows that

$$r_\nu\int_0^{r_\nu}(t^{-2}-r_\nu^{-2})N(t)dt\geq(1+o(1))N(r_\nu)\int_{r'_\nu}^{r_\nu}(t^{-2}-r_\nu^{-2})t^\mu r_\nu^{-\mu+1}dt$$

for all sufficiently large ν. Hence if $\mu>1$,

$$r_\nu\int_0^{r_\nu}(t^{-2}-r_\nu^{-2})N(t)dt\geq\frac{2(1+o(1))}{\mu^2-1}N(r_\nu),$$

so that the inequality of Lemma 1 yields

$$4T(r_\nu,f)\geq\ \{2+2\cos\gamma+\frac{2(1+o(1))\cos\gamma}{\mu^2-1}\}N(r_\nu)+Cr_\nu.$$

Since $T(r, f)/r$ grows to infinity with r, we thus obtain

$$2 \geq \{\frac{\mu^2-1+\mu^2\cos\gamma}{\mu^2-1}+o(1)\}\frac{N(r_\nu)}{T(r_\nu)}+o(1)$$

for all sufficiently large ν. The assertion now follows.

For the case $\mu=1$,

$$r_\nu\int_0^{r_\nu}(t^{-2}-r_\nu^{-2})N(t)dt \geq (1+o(1))\ \{\log(r_\nu/r'_\nu)-1/2+o(1)\}\ N(r_\nu).$$

Combining this information with the inequality of Lemma 1 we obtain

$$4T(r_\nu, f) \geq (1+o(1))\ \{\log(r_\nu/r'_\nu)+O(1)\}\ N(r_\nu)\cos\gamma + Cr_\nu$$

for all sufficiently large ν. Accordingly $N(r_\nu)/T(r_\nu)$ converges to 0 as ν tends to infinity, and hence the assertion also follows. The proof of Lemma 2 is now complete.

We shall require the following elementary result on meromorphic functions of genus zero.

Lemma 3. *Let $f(z)$ be meromorphic with $f(0)=1$. Assume that its genus is zero, and that all its zeros $\{a_\nu\}$ and poles $\{b_\nu\}$ satisfy*

$$|\pi-\arg a_\nu| \leq \gamma < \pi/2, \quad |\arg b_\nu| \leq \gamma < \pi/2$$

respectively. Then

$$4rT(r, f) \geq 2(1-\cos\gamma)rN(r)+\cos\gamma\int_0^r N(t)dt + r^2\cos\gamma\int_r^{+\infty} t^{-2}N(t)dt$$

for all values of $r>0$, where $N(r)=N(r, 0, f)+N(r, f)$.

Proof. It is clear that

$$\int_0^{2\pi}\log|1-re^{i(t-s)}|\cos\ tdt = -\sum_{n\geq 1}(1/n)\int_0^{2\pi} r^n\cos n(t-s)\cos\ tdt$$

$$= -\pi r\cos\ s$$

for $0 \leq r < 1$ and any real s. Hence for $r>1$ and any real s,

$$\int_0^{2\pi} \log |1-re^{i(t-s)}| \cos tdt = \int_0^{2\pi} \log |(1/r)e^{-i(t-s)}-1| \cos tdt$$

$$= -\pi (1/r)\cos s.$$

It therefore follows that

$$\sum_\nu \int_0^{2\pi} \log |1-(re^{it}/a_\nu)| \cos tdt$$

$$= -\sum_{|a_\nu|>r} \frac{\pi r\cos\alpha_\nu}{|a_\nu|} - \sum_{|a_\nu|\le r} \frac{\pi |a_\nu| \cos\alpha_\nu}{r}$$

for $r>0$, where α_ν denote the arguments of zeros a_ν. By the assumption $|\pi-\alpha_\nu|\le\gamma$, $\cos\alpha_\nu < -\cos\gamma$ for all ν. Hereby

$$\sum_\nu \int_0^{2\pi} \log |1-(re^{it}/a_\nu)| \cos tdt$$

$$\geq \{ \sum_{|a_\nu|>r} \frac{\pi r}{|a_\nu|} + \sum_{|a_\nu|\le r} \frac{\pi |a_\nu|}{r} \} \cos\gamma$$

$$= \pi \{ \int_0^r (t/r)dn(t,0) + \int_r^{+\infty} (r/t)dn(t,0)\} \cos\gamma$$

for $r>0$. Similarly, for the poles $\{b_\nu\}$, we have

$$\sum_\nu \int_0^{2\pi} \log |1-(re^{it}/b_\nu)| \cos tdt$$

$$= -\sum_{|b_\nu|>r} \frac{\pi r \cos\beta_\nu}{|b_\nu|} - \sum_{|b_\nu|\le r} \frac{\pi |b_\nu| \cos\beta_\nu}{r}$$

for $r>0$, where β_ν indicate the arguments of poles b_ν. By the assumption $|\beta_\nu|\le\gamma$, $\cos\beta_\nu > \cos\gamma$ for all ν, so that

$$\sum_\nu \int_0^{2\pi} \log |1-(re^{it}/b_\nu)| \cos tdt$$

$$\leq -\{ \sum_{|b_\nu|>r} \frac{\pi r}{|b_\nu|} + \sum_{|b_\nu|\le r} \frac{\pi |b_\nu|}{r} \}\cos\gamma$$

$$= -\pi \{ \int_0^r (t/r)dn(t,\infty) + \int_r^{+\infty} (r/t)dn(t,\infty)\} \cos\gamma$$

for $r>0$. Since the genus of $f(z)$ is zero and $f(0)=1$, we thus obtain

$$\int_0^{2\pi} \log |f(re^{it})| \cos tdt \geq \pi \{ \int_0^r (t/r)dn(t) + \int_r^{+\infty} (r/t)dn(t)\} \cos\gamma ,$$

where $n(t)=n(t,\ 0)+n(t,\ \infty)$. An integration by parts yields

$$\int_0^r (t/r)dn(t)=n(r)-\int_0^r r^{-1}n(t)dt=n(r)-N(r)+\int_0^r r^{-1}N(t)dt$$

for $r>0$, and since $n(r)/r$ and $N(r)/r$ both converge to 0 as r tends to infinity, we also have

$$\int_r^{+\infty}(r/t)dn(t)=\int_r^{+\infty}rt^{-2}n(t)dt-n(r)=\int_r^{+\infty}rt^{-2}N(t)dt-N(r)-n(r)$$

for $r>0$. Consequently, as in the proof of Lemma 1,

$$\begin{aligned}2\pi\{2T(r,f)-N(r)\} &=\int_0^{2\pi}|\log|f(re^{it})||dt\\ &\geq\int_0^{2\pi}\log|f(re^{it})|\cos t dt\\ &\geq\pi\left\{\int_0^r r^{-1}N(t)dt+\int_r^{+\infty}rt^{-2}N(t)dt-2N(r)\right\}\cos\gamma\end{aligned}$$

for all real values of $r>0$, which completes the proof.

2. Proof of Theorem 1. We may assume without loss of generality that $f(0)=1$ and that the order of the counting function $N(r)=N(r,0,f)+N(r,f)$ is equal to λ. Then this function $N(r)$ has a sequence $\{r_\nu\}$ of Polya peaks of the second kind of order λ. Let $\{r'_\nu\}$ and $\{r''_\nu\}$ be the associated sequences such that $\{r'_\nu\}$, $\{r_\nu/r'_\nu\}$ and $\{r''_\nu/r_\nu\}$ converge to infinity with ν, and that

$$N(t)\geq(1+o(1))(t/r_\nu)^\lambda N(r_\nu)$$

for $r'_\nu\leq t\leq r''_\nu$. With the help of Lemma 3 we obtain

$$4T(r_\nu,f)-2(1-\cos\gamma)N(r_\nu)$$

$$\geq(1+o(1))\left\{\int_{r'_\nu}^{r_\nu}t^\lambda r_\nu^{-\lambda-1}dt+\int_{r_\nu}^{r''_\nu}t^{\lambda-2}r_\nu^{1-\lambda}dt\right\}N(r_\nu)\cos\gamma$$

for all sufficiently large ν. Therefore if $\lambda<1$,

$$4T(r_\nu, f) - 2(1-\cos\gamma)N(r_\nu)$$
$$\geq (1+o(1))\ \{\frac{1-(r'_\nu/r_\nu)^{\lambda+1}}{\lambda+1} + \frac{(r''_\nu/r_\nu)^{\lambda-1}-1}{\lambda-1}\}N(r_\nu)\cos\gamma$$
$$= (1+o(1))\ \{\frac{1-o(1)}{\lambda+1} + \frac{1-o(1)}{1-\lambda}\}N(r_\nu)\cos\gamma\ ,$$

so that

$$4T(r_\nu, f) - 2(1-\cos\gamma)N(r_\nu) \geq \frac{2+o(1)}{1-\lambda^2}N(r_\nu)\cos\gamma\ .$$

This implies the desired inequality. If $\lambda = 1$,

$$4T(r_\nu, f) - 2(1-\cos\gamma)N(r_\nu)$$
$$\geq (1+o(1))\ \{\frac{1-o(1)}{\lambda+1} + \log(r''_\nu/r_\nu)\}\ N(r_\nu)\cos\gamma$$

for all sufficiently large ν. It therefore follows that

$$0 \leq \liminf_{r\to\infty} \frac{N(r)}{T(r,f)} \leq \lim_{\nu\to\infty} \frac{N(r_\nu)}{T(r_\nu, f)} = 0,$$

which completes the proof of Theorem 1.

3. Proof of Theorem 2. We may assume without loss of generality that $f(0)=1$. If the zeros $\{a_\nu\}$ and poles $\{b_\nu\}$ satisfy

$$\sum_\nu |a_\nu|^{-q} + \sum_\nu |b_\nu|^{-q} < +\infty,$$

then by the definition of the counting function $N(r) = N(r,0,f) + N(r,f)$, the integral $\int_0^{+\infty} t^{-q-1}N(t)dt$ converges. Hence $N(r)/r^q$ tends to 0 as r does to infinity. Therefore since the genus of $f(z)$ is q, $T(r,f)/r^q$ converges to some positive value as r tends to infinity. Thereby $\lambda = \mu = q$, and $N(r)/T(r,f)$ converges to 0 as r tends to infinity.

Henceforth we assume that

$$\sum_\nu |a_\nu|^{-q} + \sum_\nu |b_\nu|^{-q} = +\infty.$$

Let p be $2[(q-1)/2]+1$. Then $p=q$ if q is odd, and $p=q-1$ when q is even. In any case p is a positive odd integer and $p \leq q$. Hence

$$\sum_\nu |a_\nu|^{-p}+\sum_\nu |b_\nu|^{-p}=+\infty,$$

and by the assumption $2q\gamma<\pi$, $|\pi-\arg a_\nu^p|\leq p\gamma<\pi/2$ and $|\arg b_\nu^p|\leq p\gamma<\pi/2$ for all ν. Now we consider the auxiliary function $g(z)$ defined by

$$g(z)=f(\omega\zeta)f(\omega^2\zeta)f(\omega^3\zeta)\cdots f(\omega^p\zeta),$$

where $\zeta=z^{1/p}$ and ω is the pth root of unity, that is, $\omega=\exp(2\pi i/p)$. Then $g(z)$ is single-valued and meromorphic in the whole finite plane. Clearly $g(0)=1$, and all the zeros and poles of $g(z)$ are $\{a_\nu^p\}$ and $\{b_\nu^p\}$, respectively, so that $N(r^p,0,g)=pN(r,0,f)$ and $N(r^p,g)=pN(r,f)$ for $r>0$. Furthermore it follows from the definition of $g(z)$ that $m(r^p,g)\leq pm(r,f)$, so that $T(r^p,g)\leq pT(r,f)$ for $r>0$. In particular the lower orders $\mu(f)$ and $\mu(g)$ satisfy $p\mu(g)\leq\mu(f)$. By these facts we can apply Lemma 2 to this function $g(z)$. Accordingly the lower order $\mu(g)$ of $g(z)$ satisfies $\mu(g)\geq 1$, and

$$\liminf_{r\to\infty}\frac{N(r,0,g)+N(r,g)}{T(r,g)}\leq\frac{2(\mu(g)^2-1)}{\mu(g)^2-1+\mu(g)^2\cos p\gamma}.$$

Hence the lower order $\mu(f)$ of $f(z)$ satisfies $\mu(f)\geq p\mu(g)\geq p$, and further

$$\liminf_{r\to\infty}\frac{N(r,0,f)+N(r,f)}{T(r,f)}\leq\frac{2(\mu(g)^2-1)}{\mu(g)^2-1+\mu(g)^2\cos p\gamma}.$$

Since the real rational function $2(x^2-1)/(x^2-1+x^2\cos p\gamma)$ is increasing for $x\geq 0$, we obtain

$$\liminf_{r\to\infty}\frac{N(r,0,f)+N(r,f)}{T(r,f)}\leq\frac{2(\mu(f)^2-p^2)}{\mu(f)^2-p^2+\mu(f)^2\cos p\gamma},$$

which is the former estimation with the lower order μ of $f(z)$.

Next we shall show the latter estimation

$$\liminf_{r\to\infty}\frac{N(r,0,f)+N(r,f)}{T(r,f)}\leq\frac{2(s^2-\lambda^2)}{s^2-\lambda^2+\lambda^2\cos s\gamma},$$

where $s=2[(q+1)/2]+1$ and λ is the order of $f(z)$. The integer s is odd, and $s=q+1$ or $s=q+2$ according to whether q is even or odd.

Especially $s \geq q+1 \geq \lambda$ and $s\gamma \leq (q+2)\gamma < \pi/2$, so the right-hand side of the above estimation is zero or positive. If the order of the counting function $N(r)=N(r,0,f)+N(r,f)$ is less than λ, then the left-hand side of the above estimation is clearly zero. Hence there is nothing to prove. We therefore assume that the counting function $N(r)$ has order λ. Since $s \geq q+1$ and the genus of the function $f(z)$ is q,

$$\sum_\nu |a_\nu|^{-s} + \sum_\nu |b_\nu|^{-s} < +\infty.$$

As before, using this odd integer s, we define the function

$$h(z)=f(\sigma\zeta)f(\sigma^2\zeta)f(\sigma^3\zeta)\cdots f(\sigma^3\zeta),$$

where $\zeta = z^{1/s}$ and $\sigma = \exp(2\pi i/s)$. Then this function $h(z)$ is single-valued and meromorphic in the whole finite plane, whose zeros and poles are $\{a_\nu{}^s\}$ and $\{b_\nu{}^s\}$,respectively. Since s is an odd interger and $s \leq q+2$, it follows that $|\pi - \arg a_\nu{}^s| \leq s\gamma$ and $|\arg b_\nu{}^s| \leq s\gamma$ for $\nu \geq 1$, and the bound $s\gamma$ satisfies $0 \leq s\gamma \leq (q+2)\gamma < \pi/2$. Furthermore $N(r^s,0,h)=sN(r,0,f)$, $N(r^s,h)=sN(r,f)$ and $m(r^s,h) \leq sm(r,f)$ for $r>0$. It thereby follows that $T(r^s,h) \leq sT(r,f)$ for $r>0$, so that the order $\lambda(h)$ of the function $h(z)$ satisfies $\lambda(h) \leq \lambda(f)/s = \lambda/s$. Since $s \geq q+1 \geq \lambda$, $\lambda(h) \leq 1$, and hence the genus of $h(z)$ is 0 or 1.

If $h(z)$ has genus 1, then

$$\lim_{r\to\infty} \frac{N(r,0,h)+N(r,h)}{T(r,h)} = 0.$$

Indeed, since the counting function $N(r)=N(r,0,h)+N(r,h)$ satisfies

$$\int_0^{+\infty} t^{-2} N(t)dt < +\infty,$$

the canonical product of the zeros of $h(z)$ and that of the poles of $h(z)$ are both of genus zero. Hereby $T(r,h)/r$ converges to some positive value, while $N(r)/r$ converges to zero when r tends to infinity. Evidently

$$\liminf_{r\to\infty} \frac{N(r,0,f)+N(r,f)}{T(r,f)} = 0,$$

so that the desired estimation is proved in this case.

If $h(z)$ is of genus zero, we can apply Theorem 1 to this function $h(z)$ and have

$$\liminf_{r\to\infty}\frac{N(r,0,h)+N(r,h)}{T(r,h)}\leq\frac{2(1-\lambda(h)^2)}{1-\lambda(h)^2+\lambda(h)^2\cos s\gamma},$$

so that

$$\liminf_{r\to\infty}\frac{N(r,0,f)+N(r,f)}{T(r,f)}\leq\frac{2(1-\lambda(h)^2)}{1-\lambda(h)^2+\lambda(h)^2\cos s\gamma}.$$

Since $h(z)$ is of genus zero, the order $\lambda(h)$ is equal to that of the counting function $N(r,0,h)+N(r,h)$. On the other hand the order $\lambda(f)$ of $f(z)$ is equal to that of $N(r,0,f)+N(r,f)$ by assumption. Consequently $s\lambda(h)=\lambda(f)$, and hence the latter estimation follows. This completes the proof.

4. Proof of Corollary 1. Let λ and μ denote the order and the lower order of the function $f(z)$, respectively. We first consider the case where $f(z)$ has genus zero. Evidently $0\leq\mu\leq\lambda\leq 1$ in this case. Hence by assumption, the order λ must be 1. It thus follows from Theorem 1 that

$$\liminf_{r\to\infty}\frac{N(r,0,f)+N(r,f)}{T(r,f)}=0.$$

We next consider the case where the genus q of the function $f(z)$ is a positive even integer. In this case $p=2[(q-1)/2]+1$ is equal to $q-1$, and $s=2[(q+1)/2]+1$ is $q+1$. Hence Theorem 2 yields $q-1=p\leq\mu\leq\lambda\leq q+1=s$, so $\mu=p$ or $\lambda=s$ by assumption. Therefore we also have the above estimation by means of Theorem 2 again.

Finally we assume that the genus q is a positive odd integer. Then $p=2[(q-1)/2]+1$ is q, and $s=2[(q+1)/2]+1$ is $q+2$, and Theorem 2 yields $q=p\leq\mu\leq\lambda\leq q+2=s$. Hence $\mu=p$ *or* $\lambda=s$ again, so that we also obtain the above estimation. This completes the proof.

5. Examples. Let q be a positive even integer and let λ, μ be arbitrary real numbers with $q-1<\mu<q<\lambda<q+1$. Hereafter we construct a meromorphic function with negative zeros and positive poles, whose order is exactly λ and whose lower order is not greater

than μ.

Let $E(z,q)$ be the Weierstrass primary factor of genus q. Then

$$\begin{aligned}\log\{E(-z,q)/E(z,q)\} &= \log\{(1+z)/(1-z)\} + \sum_{n=1}^{q} \{(-1)^n - 1\}\, z^n/\, n \\ &= \log\{(1+z)/(1-z)\} - 2\sum_{n=1}^{k} z^{2n-1}/(2n-1) \\ &= 2\sum_{n=k+1}^{+\infty} z^{2n-1}/(2n-1),\end{aligned}$$

where $k=q/2$. Hence in the range $|z|\geq 2$

$$\begin{aligned}\log|E(-z,q)/E(z,q)| &\leq \log(3|z|/2) + 2\sum_{n=1}^{k} |z|^{2n-1}/(2n-1) \\ &\leq \log(3|z|/2) + q|z|^{q-1}.\end{aligned}$$

Therefore we can find a positive constant A_q such that

$$\log|E(-z,q)/E(z,q)| \leq A_q|z|^{q-1}$$

for all values of z with $|z|\geq 2$. In the range $|z|\leq 1/2$

$$\log|E(-z,q)/E(z,q)| \leq 2\sum_{n=k+1}^{+\infty} |z|^{2n-1}/(2n-1) \leq \frac{2}{q+1}\sum_{n=k+1}^{+\infty} |z|^{2n-1}.$$

It therefore follows that

$$\log|E(-z,q)/E(z,q)| \leq B_q|z|^{q+1}$$

for all values of z with $|z|\leq 1/2$, where B_q is a positive constant.

Let m be a positive integer satisfying

$$m \geq \frac{(q+1-\mu)\lambda}{(\mu-q+1)(q+1-\lambda)}.$$

For convenience we set $\eta=\mu-q+1$. Since $q-1<\mu<q$, η is positive and less than one. Since $q<\lambda<q+1$ and the positive integer m satisfies $\eta(q+1-\lambda)m\geq(2-\eta)\lambda$, it follows that $\eta m>\lambda$.

Let z_1 be an arbitrary integer with $z_1\geq 2$, and let z_ν be the integer $z_{\nu-1}{}^m$ for $\nu\geq 2$. Then $\{z_\nu\}$ is a strictly increasing sequence of positive integers. Let $\{a_j\}$ be the sequence of positive real numbers defined with $a_j = z_1^{1/\lambda}$ for $1 \leq j \leq z_1$ and $a_j = z_\nu^{1/\lambda}$ for $z_{\nu-1} < j \leq z_\nu (\nu \geq 2)$. Evidently this sequence increases without bound. Let $n(t)$ stand for the counting function of this sequence $\{a_j\}$. It is clear by definition that $n(t) = z_\nu$ for $z_\nu^{1/\lambda} \leq t < z_{\nu+1}^{1/\lambda}$. Since $z_{\nu+1} = z_\nu^m$,

$$(\lambda/m)\log t < \log n(t) \leq \lambda \log t$$

for $t \geq z_1^{1/\lambda}$. Hence the order of $n(t)$ is equal to λ, so that

$$\sum_{j\geq 1} |a_j|^{-q} = +\infty, \qquad \sum_{j\geq 1} |a_j|^{-q-1} < +\infty.$$

Hereby we can consider the meromorphic function

$$f(z) = \prod_{j\geq 1} \frac{E(-z/a_j, q)}{E(z/a_j, q)}$$

in the whole finite plane. The zeros and poles of this function $f(z)$ are $\{-a_j\}$ and $\{a_j\}$, respectively. Furthermore its order is λ exactly, and its genus is equal to the even q.

Let r_ν denote the positive number $z_\nu^{1/\eta}$ for $\nu \geq 1$. Then we can claim that

$$T(r_\nu, f) \leq O(r_\nu^{\ \mu}).$$

Since $1/\eta - 1/\lambda > 0$ and $m/\lambda - 1/\eta > 0$, $4z_\nu^{1/\lambda} < 2r_\nu < z_{\nu+1}^{1/\lambda}$ for all sufficiently large ν. Therefore $a_j \leq r_\nu$ implies $a_j \leq z_\nu^{1/\lambda}$, so that $r_\nu/a_j > 2$. Similarly if $a_j > r_\nu$, then $a_j \geq z_{\nu+1}^{1/\lambda}$, and hence $r_\nu/a_j < 1/2$. It thus follows that for all sufficiently large ν

$$\log|f(z)| = \sum_{j\geq 1} \log|E(-z/a_j, q)/E(z/a_j, q)|$$

$$\leq \sum_{a_j \leq r_\nu} A_q |z/a_j|^{q-1} + \sum_{a_j > r_\nu} B_q |z/a_j|^{q+1}$$

on the circle $|z| = r_\nu$. Since $a_j > 1$, $n(r_\nu) = z_\nu$, $z_\nu = r_\nu^{\ \eta}$ and $\eta = \mu - q + 1$,

$$\sum_{a_j \leq r_\nu} A_q |z/a_j|^{q-1} \leq A_q |z|^{q-1} n(r_\nu) = A_q r_\nu^{\ q-1+\eta} = A_q r_\nu^{\ \mu}$$

for $|z| = r_\nu$. Next we estimate the second term

$$\sum_{a_j>r_\nu} B_q \mid z/a_j \mid^{q+1} = B_q \mid z \mid^{q+1} \sum_{a_j>r_\nu} \mid a_j \mid^{-q-1}.$$

By the definition of the sequence $\{a_j\}$, $a_j>r_\nu$ implies $a_j=z_n^{1/\lambda}$ for some n with $n>\nu$, and the multiplicity of the point $z_n^{1/\lambda}$ is z_n-z_{n-1}. Hence

$$\sum_{a_j>r_\nu} \mid a_j \mid^{-q-1} = \sum_{n>\nu} (z_n-z_{n-1}) z_n^{-(q+1)/\lambda} \leq \sum_{n>\nu} z_n^{\tau},$$

where $\tau=1-(q+1)/\lambda$. Since $\log z_n=m^{n-\nu}\log z_\nu$ and $m^{n-\nu}\geq(n-\nu)m$ for $n>\nu$, we have $z_n>z_\nu^{(n-\nu)m}$. Furthermore since $\tau<0$, $z_n^{\tau}<z_\nu^{(n-\nu)m\tau}$ for $n>\nu$. It thus follows that

$$\sum_{a_j>r_\nu} \mid a_j \mid^{-q-1} \leq \sum_{n>\nu} z_n^{\tau} \leq \sum_{j\geq 1} (z_\nu^{m\tau})^j = \frac{z_\nu^{m\tau}}{1-z_\nu^{m\tau}} \leq 2z_\nu^{m\tau}$$

for all sufficiently large ν. Consequently, on the circle $|z|=r_\nu$

$$\log \mid f(z) \mid \leq A_q r_\nu^{\mu} + 2B_q r_\nu^{q+1} z_\nu^{m\tau}$$

$$\leq A_q r_\nu^{\mu} + 2B_q r_\nu^{q+1+m\tau\eta},$$

so that by definition

$$m(r_\nu, f) \leq A_q r_\nu^{\mu} + 2B_q r_\nu^{q+1+m\tau\eta}$$

for all sufficiently large ν. Note that

$$q+1+m\tau\eta = q+1-m(q+1-\lambda)(\mu-q+1)/\lambda \leq \mu.$$

Then we obtain $m(r_\nu, f)\leq O(r_\nu^{\mu})$ for all sufficiently large ν. On the other hand it is clear by definition that

$$N(r_\nu, f) = \sum_{a_j\leq r_\nu} \log \frac{r_\nu}{a_j} \leq n(r_\nu)\log r_\nu = z_\nu \log r_\nu = r_\nu^{\eta} \log r_\nu$$

for all ν. Since $\mu>\eta$, $N(r_\nu, f)=O(r_\nu^{\mu})$ for all sufficiently large ν. We therefore have $T(r_\nu, f)=O(r_\nu^{\mu})$, as we claimed. Accordingly the lower order of $f(z)$ does not exceed μ.

Again let q be an arbitrary positive even integer, and let $\{a_\nu\}$ be the sequence defined with $a_\nu=\nu^{1/q}(\nu\geq 1)$. By $n(t)$ we denote the counting function of this sequence. Evidently $n(t)=\nu$ for $\nu^{1/q}\leq t<(\nu+1)^{1/q}$, so that $0\leq t^q-n(t)<1$ for $t\geq 0$. Therefore the order of this sequence is precisely equal to q, and the integral $\int_0^{+\infty} t^{-q-2}\, n(t)dt$ is surely convergent. Hence we can define the meromorphic function

$$f(z)=\Pi_{\nu\geq 1}\frac{E(-z/a_\nu,q)}{E(z/a_\nu,q)}$$

in the whole finite plane. This function $f(z)$ has only negative zeros and positive poles. Since the genus q is even, we have the integral representation

$$\begin{aligned}\log f(z)&=\sum_{\nu\geq 1}\log\{E(-z/a_\nu,q)/E(z/a_\nu,q)\}\\&=\int_0^{+\infty}\log\ \{E(-z/t,q)/E(z/t,q)\}\ dn(t)\\&=2z^{q+1}\int_0^{+\infty}\frac{n(t)}{(t^2-z^2)t^q}dt\end{aligned}$$

in the upper half plane, where the branch of the logarithm is taken so that $\log f(0)=0$. By means of contour integrals and residue theory,

$$\int_0^{+\infty}\frac{2z}{t^2-z^2}dt=i\pi$$

for any values of $z=x+iy$ with $y>0$. Accordingly

$$\log f(z)-i\pi z^q=2z^{q+1}\int_0^{+\infty}\frac{n(t)-t^q}{(t^2-z^2)t^q}dt,$$

so that

$$|\log f(z)-i\pi z^q|\leq 2|z|^{q+1}\int_0^{+\infty}\frac{t^q-n(t)}{|t^2-z^2|t^q}dt$$

in the upper half plane. Since $0\leq t^q-n(t)<1$ for $t\geq 0$ and $n(t)=0$ for $0\leq t\leq 1$,

$$|\log f(z)-i\pi z^q|\leq 3|z|^{q-1}+2|z|^{q+1}\int_1^{+\infty}\frac{1}{|t^2-z^2|t^q}dt$$

for $z=x+iy$ with $|z|\geq 2$ and $y>0$. Taking real parts we obtain

$$\left|\log|f(re^{i\theta})|+\pi r^q\sin q\theta\right|\leq 3r^{q-1}+2r^{q+1}\int_1^{+\infty}\frac{1}{|t^2-r^2e^{i2\theta}|t^q}dt$$

for $r\geq 2$ and $0<\theta<\pi$. Hereby an integration yields

$$|m(r,f)-r^q|\leq 3r^{q-1}+2r^{q+1}\int_0^{\pi}\int_1^{+\infty}\frac{1}{|t^2-r^2e^{i2\theta}|t^q}dtd\theta$$

for $r\geq 2$. By means of Schwarz's inequality and Poisson's formula,

$$\left\{\int_0^{\pi}\frac{1}{|t^2-r^2e^{i2\theta}|}d\theta\right\}^2\leq\pi\int_0^{\pi}\frac{1}{|t^2-r^2e^{i2\theta}|^2}d\theta=\frac{\pi^2}{|t^4-r^4|}$$

for any real pair t and r with $t\neq\pm r$. It therefore follows that

$$\begin{aligned}\int_1^{+\infty}\int_0^{\pi}\frac{1}{|t^2-r^2e^{i2\theta}|t^q}d\theta\,dt&\leq\int_1^{+\infty}\frac{\pi}{|t^4-r^4|^{1/2}t^q}dt\\&=\pi r^{-q-1}\int_{1/r}^{+\infty}\frac{1}{|1-s^4|^{1/2}s^q}ds,\end{aligned}$$

so that for any real $r>2$

$$\begin{aligned}\int_1^{+\infty}\int_0^{\pi}\frac{1}{|t^2-r^2e^{i2\theta}|t^q}d\theta\,dt&\leq Cr^{-q-1}+2\pi r^{-q-1}\int_{1/r}^{1/2}s^{-q}ds\\&\leq Cr^{-q-1}+\frac{2\pi}{(q-1)r^2}\end{aligned}$$

with a positive constant C. Interchanging the order of the integrations we thus have

$$|m(r,f)-r^q|\leq 3r^{q-1}+2C+Dr^{q-1}$$

with a positive constant D. Consequently

$$|m(r,f)-r^q|\leq O(r^{q-1}),$$

and hence $m(r,f)/r^q$ converges to 1 as r tends to infinity. On the other hand it is clear that $N(r)=N(r,f,0)=N(r,f)$ and

$$0\leq\int_1^r t^{q-1}dt-N(r)\leq\log r$$

for $r>0$. Hence $N(r)/r^q$ converges to $1/q$ when r tends to infinity. We consequently have

$$\lim_{r\to\infty}\frac{T(r,f)}{r^q}=\frac{q+1}{q},$$

so that the order of the function $f(z)$ is equal to the even integer q. Furthermore it follows that

$$\lim_{r\to\infty}\frac{N(r,0,f)+N(r,f)}{T(r,f)}=\frac{2}{q+1}.$$

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