Factorization of Periodic Entire Functions

By Tadashi Kobayashi

We say that an entire function $F(z) = f(g(z))$ has $f(z)$ and $g(z)$ as left and right factors respectively, provided that $f(z)$ and $g(z)$ are both meromorphic functions. Such a composition $f(g(z))$ is called a factorization of $F(z)$. An entire function $F(z)$ is said to be prime if every factorization of the above form implies that either $f(z)$ or $g(z)$ is a linear function.

In the paper [4] Gross posed an open question whether there exists a periodic entire function which is prime. This question was resolved by Ozawa [8] positively. In fact he succeeded to construct a prime periodic entire function of order one. Subsequently several interesting such entire functions have been found [2], [9], [12].

The purpose of this paper is to present a method by which prime and periodic entire functions can be constructed. Our method is very elementary in principle. In what follows we assume acquaintance with the standard terminology and the fundamental concepts of Nevanlinna theory, and we use them without further introduction.

1. Statement of results

Let $\{a_n\}$ and $\{b_n\}$ be sequences of distinct complex numbers, tending to infinity, such that $a_n \neq 0$, $b_n \neq 0$ and $a_m \neq b_n$ for any m and n. Let $\{p_n\}$ and $\{q_n\}$ be sequences of distinct prime numbers such that $p_n \geqq 3$, $q_n \geqq 3$ and $p_m \neq q_n$ for any m and n. Now assume that the series

$$(1.1) \qquad \sum_{n\geqq 1} p_n(\log|a_n|)^{-\alpha}, \qquad \sum_{n\geqq 1} q_n(\log|b_n|)^{-\beta}$$

are both convergent for some positive numbers α and β. Then it is clear that the series $\sum p_n|a_n|^{-1}$ and $\sum q_n|b_n|^{-1}$ are also convergent.

Therefore the canonical products

$$(1.2) \qquad A(z) = \Pi_{n\geqq 1} \left(1-\frac{z}{a_n}\right)^{p_n},$$

$$(1.3) \qquad B(z) = \Pi_{n\geqq 1} \left(1-\frac{z}{b_n}\right)^{q_n}$$

both converge and represent transcendental entire functions. The former function $A(z)$ vanishes only at the points a_n, and their multiplicities are equal to the prime numbers p_n, respectively. It follows from the definition (1.2) that

$$(1.4) \qquad \begin{aligned} \log|A(z)| &\leqq \textstyle\sum_{n\geqq 1} p_n \log(1 + \frac{r}{|a_n|}) \\ &= \int_0^\infty \log(1 + \frac{r}{t}) dn(t) \\ &= r\int_0^\infty \frac{n(t)}{t(t+r)}\, dt \end{aligned}$$

for all values of z with $|z| = r$, where $n(t) = n(t, 0, A)$. It also follows from (1.1) that

$$(1.5) \qquad n(r, 0, A)(\log r)^{-\alpha} \leqq \textstyle\sum_{n\geqq 1} p_n (\log|a_n|)^{-\alpha}$$

for $r>0$. Combining this (1.5) with (1.4) we thus deduce

$$(1.6) \qquad \log M(r, A) = O\,\{(\log r)^{1+\alpha}\}$$

as r tends to infinity, where $M(r, A)$ stands for the maximum modulus of $A(z)$ on $|z| = r$. Similarly the entire function $B(z)$ vanishes only at the points b_n, and the multiplicities of b_n are equal to the prime numbers q_n, respectively. With the help of (1.1) and (1.3) we can also have

$$(1.7) \qquad \log M(r, B) = O\,\{(\log r)^{1+\beta}\}$$

as r tends to infinity.

Let $a(z)$ and $b(z)$ be nonconstant rational functions which are regular in $0<|z|<+\infty$. Assume that $a(z)$ or $b(z)$ has a pole at the point at infinity, and that $a(z)$ or $b(z)$ has a pole at the origin. Assume further that for any n, the equations $a(z) = a_n$ and $b(z) = b_n$ have no multiple roots, and that for any pair m and n, the equations $a(z) = a_m$ and $b(z) = b_n$ have no common solutions.

Under these notations and assumptions we consider the function

$$F(z) = A(a(e^z))B(b(e^z)). \tag{1.8}$$

Since the rational functions $a(z)$ and $b(z)$ are regular in $0<|z|<+\infty$, the functions $a(e^z)$ and $b(e^z)$ are both regular in the whole finite plane. Hence the above function $F(z)$ is entire and periodic with period $2\pi i$. Furthermore since $a(e^z)$ and $b(e^z)$ are of exponential type, by means of (1.6) and (1.7), the function $F(z)$ must be of finite order. By the above assumption that either $a(z)$ or $b(z)$ has a pole at the origin, either $A(a(z))$ or $B(b(z))$ has an essential singularity at there. In fact if $a(z)$ has a pole at the origin, it is possible to find a sequence $\{z_n\}$ tending to zero such that $a(z_n) = a_n$, because the sequence $\{a_n\}$ tends to infinity. It is clear by the definition (1.2) that all the points z_n are zeros of the composite function $A(a(z))$ which is not constant zero. Hereby $A(a(z))$ has an essential singularity at the origin. Consequently the product $A(a(z))$ and $B(b(z))$ is certainly not meromorphic in the whole finite plane.

Our results are the following

Theorem 1. *Assume that the function $F(z)$ admits a factorization $f(g(z))$ with nonlinear entire $f(z)$ and transcendental entire $g(z)$. Then the right factor $g(z)$ is periodic and of exponential type, and there exist two polynomials $P(z)$ and $Q(z)$ such that $a(e^z) = P(g(z))$ and $b(e^z) = Q(g(z))$. Furthermore the rational functions $a(z)$ and $b(z)$ both have poles at the point at infinity and at the original point simultaneously.*

Theorem 2. *Assume that the function $F(z)$ admits a factorization $f(g(z))$ with transcendental entire $f(z)$ and nonlinear polynomial $g(z)$. Then the rational functions $a(z)$ and $b(z)$ satisfy $a(z) = a(\omega/z)$ and $b(z) = b(\omega/z)$ with a nonzero constant ω.*

Theorem 3. *Assume that the function $F(z)$ is prime in the family of entire functions. Then $F(z)$ is prime.*

Assume that the function $F(z)$ is not prime. Then by virtue of Theorem 3, $F(z)$ admits a nontrivial factorization $F(z)=f(g(z))$ with entire functions $f(z)$ and $g(z)$. Assume temporarily that the right factor $g(z)$ is polynomial. Then by means of Theorem 2, the rational functions $a(z)$ and $b(z)$ satisfy $a(z)=a(\omega/z)$ and $b(z)=b(\omega/z)$ for some nonzero constant ω. It therefore follows that

$$a(z)=\sum_{j=0}^{m}\alpha_j\{z^j+(\omega/z)^j\}, \qquad b(z)=\sum_{j=0}^{n}\beta_j\{z^j+(\omega/z)^j\},$$

where $m>0$, $n>0$ and α_j, β_j are constants with $\alpha_m\neq 0$, $\beta_n\neq 0$. In particular, $a(z)$ and $b(z)$ both have poles at the origin and at the point at infinity, simultaneously. On the other hand it is well known that for each nonnegative integer k, there exists a polynomial $C_k(z)$ of degree k which satisfies $C_k(z+\omega/z)=z^k+(\omega/z)^k$. Using these polynomials $C_k(z)$ we easily deduce

$$a(z)=\sum_{j=0}^{m}\alpha_j C_j(z+\omega/z)=P(z+\omega/z),$$

$$b(z)=\sum_{j=0}^{n}\beta_j C_j(z+\omega/z)=Q(z+\omega/z),$$

where $P(z)=\sum_{j=0}^{m}\alpha_j C_j(z)$ and $Q(z)=\sum_{j=0}^{n}\beta_j C_j(z)$, that is, $P(z)$ and $Q(z)$ are polynomials of degree m and n, respectively. It thus follows that

$$a(e^z)=P(h(z)), \qquad b(e^z)=Q(h(z)),$$

where $h(z)=e^z+\omega e^{-z}$. Evidently this function $h(z)$ is periodic and entire of exponential type. Combining this observation with Theorem 1 we at once have the following conclusion.

Theorem. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of distinct complex numbers, tending to infinity, such that $a_n\neq 0$, $b_n\neq 0$ and $a_m\neq b_n$ for any m and n. Let $\{p_n\}$ and $\{q_n\}$ be sequences of distinct prime numbers such that $p_n\geqq 3$, $q_n\geqq 3$ and $p_m\neq q_n$ for any m and n. Let $a(z)$ and $b(z)$ be nonconstant rational functions which are regular in the annulus $0<|z|<+\infty$ such that $a(z)$ or $b(z)$ has a pole at the origin, and $a(z)$ or $b(z)$ also has a pole at the point at infinity.*

Suppose that the series

$$\sum_{n\geq 1} p_n(\log|a_n|)^{-\alpha}, \qquad \sum_{n\geq 1} q_n(\log|b_n|)^{-\beta}$$

are both convergent for some positive α *and* β . *Suppose further that for any* n, *the equations* $a(z) = a_n$ *and* $b(z) = b_n$ *have no multiple roots, and that the equations* $a(z) = a_m$ *and* $b(z) = b_n$ *have no common solutions for any* m *and* n.

Let $F(z)$ *be* $A(a(e^z))B(b(e^z))$, *where* $A(z)$ *and* $B(z)$ *are the canonical products defined by*

$$A(z) = \Pi_{n\geq 1}\left(1-\frac{z}{a_n}\right)^{p_n}, \qquad B(z) = \Pi_{n\geq 1}\left(1-\frac{z}{b_n}\right)^{q_n}$$

Then $F(z)$ *is a periodic entire function of finite order, and a necessary and sufficient condition that this function* $F(z)$ *is not prime is that there exist two polynomials* $P(z)$, $Q(z)$ *and a periodic entire function* $h(z)$ *of exponential type such that* $a(e^z) = P(h(z))$ *and* $b(e^z) = Q(h(z))$.

2. Lemmas

We need several known results.

Lemma 1 [10]. *Suppose that* $f(z)$ *and* $g(z)$ *are entire functions and that the composite* $f(g(z))$ *has finite order. Then either* $f(z)$ *is of order zero or* $g(z)$ *is a polynomial.*

Lemma 2 [1], [3]. *If* $f(z)$ *is any nonconstant entire function of order less than* $1/2$ *and* $g(z)$ *is entire, then the composite function* $f(g(z))$ *is periodic if and only if the right factor* $g(z)$ *is. Furthermore the period of* $g(z)$ *is an integral multiple of that of* $f(g(z))$.

Lemma 3 [11]. *Let* $f(z)$ *be an arbitrary nonconstant entire function, and let* $g(z)$ *be an arbitrary polynomial. If the composite function* $f(g(z))$ *is periodic, then the degree of* $g(z)$ *is at most two.*

The next fact is nothing but Nevanlinna's second main theorem for entire functions of finite order.

Lemma 4 [6]. *Let* $f(z)$ *be a nonconstant entire function of finite* *order, and let* $a_1, \cdots, a_n$, *where* $n \geqq 2$, *be distinct finite complex numbers. Then*

$$\{n-1+o(1)\}\, T(r, f) \leqq \sum_{j=1}^{n} \overline{N}(r, a_j, f)$$

as r *tends to infinity.*

Lemma 5. *Let* $f(z)$ *be*

$$\sum_{k=-m}^{n} c_k \exp(kz),$$

where m *and* n *are positive integers and* c_k *are constants with* $c_{-m} \neq 0$ *and* $c_n \neq 0$. *Then* $T(r, f) \sim (m+n)r/\pi$, *and* $T(r, f) \sim N(r, w, f)$ *for any finite complex number* w.

Proof of Lemma 5. Let $R(z)$ be $\sum_{j=-m}^{n} c_j z^j$, and let $P(z)$ be $z^m R(z)$. Then it is clear that $f(z) = R(e^z)$. Let w be an arbitrary finite complex number. Since neither c_{-m} nor c_n is zero, the polynomial $P(z)-wz^m$ is of degree $m+n$ and does not vanish at $z=0$. Hence there exist $m+n$ nonvanishing finite complex numbers $s_1, s_2, \cdots, s_{m+n}$ such that

$$P(z)-wz^m = c_n \prod_{j=1}^{m+n} (z-s_j).$$

It therefore follows that

$$f(z)-w = R(e^z)-w = c_n \exp(-mz) \prod_{j=1}^{m+n} (e^z-s_j),$$

so that we easily have

$$N(r, w, f) = \sum_{j=1}^{m+n} N(r, s_j, e^z) \sim (m+n)r/\pi .$$

This completes the proof of Lemma 5.

3. Proof of Theorem 1

Let us begin by proving the following property of the function $F(z)$.

Lemma A. *There exists a sequence* $\{s_n\}$ *of zeros of* $F(z)$ *such that the real parts of* s_n *tend to positively infinite as* n *goes to infinity.*

There also exists a sequence $\{t_n\}$ of zeros of $F(z)$ such that the real parts of t_n tend to negatively infinite as n does to infinity.

Proof of Lemma A. By assumption $a(z)$ or $b(z)$ has a pole at the point at infinity. For instance suppose that $b(z)$ has a pole at the infinity. Since the zeros b_n of the canonical product $B(z)$ approach the infinity, it is possible to find a sequence $\{w_n\}$ tending to infinity such that $b(w_n) = b_n$. Here we set $s_n = \log w_n$ with a suitable choice of branch. Evidently $F(s_n) = 0$ and the real parts of s_n tend to positively infinite. This is precisely what we have to prove. Similarly we can also construct the sequence $\{t_n\}$ required in the latter statement. This completes the proof of Lemma A.

Assume that the function $F(z)$ has a nontrivial factorization

$$F(z) = f(g(z)), \tag{3.1}$$

where $f(z)$ is a nonlinear entire function and $g(z)$ is a transcendental entire function. Since $F(z)$ is of finite order and the right factor $g(z)$ is transcendental, the left factor $f(z)$ has zero order by Lemma 1. It thereby follows from Lemma 2 that the right factor $g(z)$ is periodic with period $2k\pi i$, where k is a positive integer.

By the definition (1.8), a zero of the function $F(z)$ is also a zero of $A(a(e^z))$ or $B(b(e^z))$. Hence it satisfies either $a(e^z) = a_m$ for some m, or $b(e^z) = b_n$ for some n. By the assumptions on the rational functions $a(z)$ and $b(z)$, for every j, the equations $a(e^z) = a_j$ and $b(e^z) = b_j$ have no multiple solutions. Furthermore for an arbitrary pair i and j, the equations $a(e^z) = a_i$ and $b(e^z) = b_j$ have no common solutions. Hereby if a zero of $F(z)$ satisfies $a(e^z) = a_m$, then its multiplicity must be equal to the prime number p_m. Similarly a zero of $F(z)$ satisfying $b(e^z) = b_n$ is multiple and its order is equal to the prime q_n.

Lemma B. *Let u be a simple zero of the left factor $f(z)$. Then all the roots of the equation $g(z) = u$ have multiplicities at least* 3. *Let v be a multiple zero of $f(z)$. Then all the roots of the equation $g(z) = v$ are simple and satisfy $a(e^z) = a_m$ for some m, or $b(e^z) = b_n$ for some n. In the former case the multiplicity of v is the prime number p_m, and in the latter case it is equal to the prime number q_n.*

Proof of Lemma B. Let u be a simple zero of the left factor $f(z)$, and let s be a root of the equation $g(z)=u$. Then $F(z)$ vanishes at this point s by (3.1), so that $f(g(z))$ has a zero at s with multiplicity p_m or q_n. Since the numbers p_m and q_n are prime, the point s must be a multiple root of $g(z)=u$, and its multiplicity is equal to $p_m \geqq 3$ or $q_n \geqq 3$.

Let v be a multiple zero of the left factor $f(z)$. Assume for a moment that $g(z)$ fails to take the value v. Since $F(z)$ is of finite order, $g(z)$ also has finite order. Hence $g(z)=v+\exp(P(z))$ with a polynomial $P(z)$. On the other hand $g(z)$ is periodic and its period is an integral multiple of $2\pi i$. It thus follows that $P(z)$ is linear and has the form $\alpha z+\beta$, where α is nonzero real. Thereby $g(z)$ converges to the value v when the real part of z tends to positively infinite or negatively infinite according to whether α is negative or positive. Suppose the former for definiteness. By the above Lemma A there is a sequence $\{s_n\}$ of zeros of $F(z)$ such that the real parts of s_n become positively infinite as n grows. It therefore follows that all the points $g(s_n)$ are zeros of $f(z)$ and they converge to the value v. Since $g(s_n) \neq v$ for any n, the function $f(z)$ vanishes identically. This is absurd. Consequently the equation $g(z)=v$ has at least one root. Let s be arbitrary such a root. Of course, $F(s)=f(g(s))=f(v)=0$. Hence the point s satisfies $a(e^s)=a_m$ for some m, or $b(e^s)=b_n$ for some n. In the former case $f(g(z))$ has a zero of multiplicity p_m at s. Since the zero v of $f(z)$ is multiple and the number p_m is prime, the root s of $g(z)=v$ must be simple and the multiplicity of the zero v of $f(z)$ is exactly equal to p_m. Similarly for the latter case the point s is a zero of $f(g(z))$ of multiplicity q_n. Hence the equation $g(z)=v$ has a simple root at s, and the multiplicity of the zero v of $f(z)$ is equal to the prime q_n. Lemma B is now proved.

By this Lemma B the function $f(z)$ has at least two zeros. Indeed if $f(z)$ has only one zero, then it must be simple by Lemma B. Hence $f(z)$ reduces to a linear function, because $f(z)$ is of order zero.

Let u be a simple zero of $f(z)$. Then by means of Lemma B all the roots of the equation $g(z)=u$ are multiple and their multiplicities are at least three. We hence have

$$N(r, u, g(z)) \geqq 3\, \overline{N}(r, u, g(z)) \tag{3.2}$$

for all values of $r>0$. Accordingly with the help of Lemma 4 the function $f(z)$ has at most one simple zero.

Let v be a multiple zero of $f(z)$. It then follows from Lemma B again that all the roots of $g(z)=v$ are simple and they satisfy either $a(e^z)=a_m$ for some m, or else $b(e^z)=b_n$ for some n. We thus have

$$N(r, v, g(z)) = O(r), \tag{3.3}$$

because the functions $a(e^z)$ and $b(e^z)$ are of exponential type. On taking (3.2) and (3.3) into account we consequently have $T(r, g)=O(r)$ by Lemma 4 again, that is, the function $g(z)$ is of exponential type.

Lemma C. *The right factor $g(z)$ can be written as*

$$g(z) = \sum_{j=-m}^{n} c_j \exp(jz/k), \tag{3.4}$$

where k, m and n are positive integers, and c_j $(-m \leqq j \leqq n)$ are constants with $c_{-m} \neq 0$, $c_n \neq 0$.

Proof of Lemma C. We have already pointed out that the right factor $g(z)$ is periodic and its period is an integral multiple of $2\pi i$. Hence $g(z)$ satisfies $g(z)=g(z+2k\pi i)$ with a positive integer k. Hereby the function

$$R(z) = g(k\log z) \tag{3.5}$$

is single-valued and regular in the annulus $0<|z|<+\infty$. Let γ be any finite complex number, and let $w_1, w_2, \ldots, w_l$ be any distinct l roots of the equation $R(z)=\gamma$ in $0<|z|<+\infty$. For every j, by the definition (3.5), all the roots of $\exp(z/k)=w_j$ satisfy $g(z)=R(w_j)=\gamma$. It therefore follows that

$$\sum_{j=1}^{l} N(r, w_j, \exp(z/k)) \leqq N(r, \gamma, g(z))$$

for $r>0$, so that

$$lr \leqq k\pi \ \{T(r, g(z))+O(1)\}$$

as r tends to infinity. Since $g(z)$ is of exponential type, the number l cannot exceed some finite bound. This means that for any finite complex number γ, the equation $R(z) = \gamma$ has only a finite number of roots in the annulus $0<|z|<+\infty$. Hereby $R(z)$ is a rational function. Suppose now that $R(z)$ is regular at the origin. With the help of Lemma A we can take a sequence $\{t_n\}$ of zeros of $F(z)$ such that the real parts of t_n become negatively infinite as n goes to infinity. It thus follows that $f(g(t_n)) = 0$ for any n and $\exp(t_n/k)$ converges to 0 as n tends to infinity. On the other hand by the definition (3.5), we deduce

$$g(z) = R(\exp(z/k)). \tag{3.6}$$

Hereby $g(t_n)$ converges to the finite value $R(0)$ when n tends to infinity. It thus follows that $f(z)$ vanishes identically or $R(z)$ is a constant. This is a contradiction. Accordingly the rational function $R(z)$ must have a pole at the origin. Similarly by making use of Lemma A and (3.6), we can see that $R(z)$ also has a pole at the point at infinity. Consequently $R(z)$ has the form

$$R(z) = \sum_{j=-m}^{n} c_j z^j \tag{3.7}$$

with $m>0$, $n>0$ and $c_{-m} \neq 0$, $c_n \neq 0$. Combining this (3.7) with (3.6), we deduce the desired (3.4). This completes the proof of Lemma C.

With the help of Lemma 5 and this Lemma C we at once have

$$T(r, g(z)) \sim N(r, w, g(z)) \sim (m+n)r/k\pi \tag{3.8}$$

for any finite complex number w. For the functions $a(e^z)$ and $b(e^z)$, using exactly the same argument as in the proof of Lemma 5, we can also see that

$$T(r, a(e^z)) \sim N(r, w, a(e^z)) \sim a_* r/\pi, \tag{3.9}$$

$$T(r, b(e^z)) \sim N(r, w, b(e^z)) \sim b_* r/\pi \tag{3.10}$$

for every finite complex number w with two possible exceptions, where a_* and b_* indicate the orders of $a(z)$ and $b(z)$, respectively.

We now consider the set of complex numbers

$$S = \{ a(z^k) : R'(z) = 0 \} \cup \{ b(z^k) : R'(z) = 0 \} ,$$

where $R(z)$ is the rational function defined with (3.5), and k is the positive integer appeared in (3.4). Evidently the derivative of $R(z)$ has at most $m+n$ zeros in the whole finite plane, so the above set S consists of at most $2(m+n)$ points.

Let a_μ be an arbitrary complex number of the sequence $\{a_n\}$ which is not contained in the set S and satisfies

$$T(r, a(e^z)) \sim N(r, a_\mu, a(e^z)). \tag{3.11}$$

Let s be an arbitrary root of the equation $a(e^z) = a_\mu$. Then this point s is a zero of $F(z)$ with multiplicity p_μ. Hence by (3.1), either $g(s)$ is a zero of $f(z)$ of order p_μ, or else $g(z)-g(s)$ has a multiple zero at this point s. Suppose that the latter case occurs. Then $g'(s) = 0$, so that $R'(\exp(s/k)) = 0$ by means of (3.4) and (3.6). Consequently the value $a(e^s) = a_\mu$ must be an element of the set S. This is absurd. Hereby $g'(s) \neq 0$ and the point $g(s)$ is a zero of the left factor $f(z)$ of multiplicity p_μ. Conversely let v be an arbitrary zero of $f(z)$ of order p_μ. Then by virtue of Lemma B all the roots of the equation $g(z) = v$ are simple and satisfy $a(e^z) = a_\mu$. We hence find that the set of points $\{ g(z) : a(e^z) = a_\mu \}$ surely coincides with the set of all the zeros of $f(z)$ of order exactly p_μ. Let $w_1, w_2, \ldots, w_l$ be any l zeros of $f(z)$ of order p_μ. Then it follows that

$$\sum_{j=1}^{l} N(r, w_j, g(z)) \leqq N(r, a_\mu, a(e^z))$$

for $r>0$, so that $l(m+n) \leqq a_* k$ by means of (3.8), (3.9) and (3.11). This means that the number of the zeros of $f(z)$ of order p_μ must be finite. We denote these zeros with $u_1, u_2, \ldots, u_l$, that is, the set $\{ g(z) : a(e^z) = a_\mu \}$ consists of l points $u_1, u_2, \ldots, u_l$. This time we have the equality

$$\sum_{j=1}^{l} N(r, u_j, g(z)) = N(r, a_\mu, a(e^z))$$

for $r>0$. Hereby we have $l(m+n) \leqq a_* k$ by (3.8), (3.9) and (3.11) again. Furthermore we at once have the representation

$$a(e^z)-a_\mu = \exp(\alpha z+\beta) \prod_{j=1}^{l} (g(z)-u_j) \tag{3.12}$$

with constants α and β. Because of the periodicity, $\exp(\alpha z+\beta)$ is periodic with period $2k\pi i$. Thereby αk is an integer. Inserting the representation (3.4) into this (3.12) we consequently have

$$a(z^k)-a_\mu = Cz^s \prod_{j=1}^{l} (R(z)-u_j), \tag{3.13}$$

where s is an integer and C is a nonzero constant. Since the rational function $R(z)$ has a pole of order m at the origin, if $lm-s>0$, the rational function $a(z)$ also has a pole of order $(lm-s)/k$ there. If $lm-s \leqq 0$, then $a(z)$ is regular at the origin. Similarly since $R(z)$ has a pole of order n at the point at infinity, if $ln+s>0$, $a(z)$ also has a pole of order $(ln+s)/k$ there, and it is regular there when $ln+s\leqq 0$. Here we should remark that the integral quantities l and s are both independent of the complex number a_μ.

Let a_ν be another complex number of the sequence $\{a_n\}$ which is not in the set S and satisfies

$$T(r, a(e^z)) \sim N(r, a_\nu, a(e^z)).$$

Then, just as before, the number of the zeros of $f(z)$ of multiplicity p_ν is exactly equal to $l = a_* k/(m+n)$, and we have the expression

$$a(z^k)-a_\nu = Dz^s \prod_{j=1}^{l} (R(z)-v_j), \tag{3.14}$$

where D is a nonzero constant and $v_1, v_2, \ldots, v_l$ are the zeros of $f(z)$ of order p_ν. Comparing this (3.14) with the above (3.13), we deduce

$$z^{-s}(a_\nu-a_\mu) = C \prod_{j=1}^{l} (R(z)-u_j)-D \prod_{j=1}^{l} (R(z)-v_j). \tag{3.15}$$

Evidently if the right side of this identity is not a constant, it has poles at the origin and at the infinity, while the left side is regular at the origin or at the infinity. Hereby the both sides of (3.15) must be constant and hence the exponent s must be zero. Accordingly the above (3.13) becomes

$$a(z^k)-a_\mu = C \prod_{j=1}^{l} (R(z)-u_j).$$

We consequently have $a(z^k) = P(R(z))$, where $P(z)$ is a polynomial of degree l. Setting $z = \exp(z/k)$ we at once obtain $a(e^z) = P(g(z))$ by means of (3.6). Furthermore by the form (3.7) the rational function $a(z)$ has a pole of order lm/k at the origin and has a pole of order ln/k at the point at infinity.

Quite similarly for any complex number b_μ of the sequence $\{b_n\}$ such that b_μ is not in the set S and

$$T(r, b(e^z)) \sim N(r, b_\mu, b(e^z)) \sim b_* r/\pi,$$

taking account of (3.8) and (3.10), we also have the identity

$$b(z^k)-b_\mu = C_* \prod_{j=1}^{h} (R(z)-u_j),$$

where h is the positive integer $b_* k/(m+n)$, and $u_1, u_2, \ldots, u_h$ are all the zeros of $f(z)$ of multiplicity just q_μ, and C_* is a nonzero constant. We therefore obtain the expression $b(z^k) = Q(R(z))$, where $Q(z)$ is a polynomial of degree $b_* k/(m+n)$. It is plain from (3.6) that $b(e^z) = Q(g(z))$. It is also plain from (3.7) that the rational function $b(z)$ has poles at the origin and at the point at infinity simultaneously. The proof of Theorem 1 is now complete.

4. Proof of Theorem 2

We suppose that the function $F(z)$ admits a nontrivial factorization $f(g(z))$, where $f(z)$ is a transcendental entire function and $g(z)$ is a nonlinear polynomial. Since $F(z)$ is periodic with period $2\pi i$, by virtue of Lemma 3, the right factor $g(z)$ is a quadratic polynomial. Hence we may set $g(z) = z(z+c)$ with a constant c. Obviously $g(-z-c) = z(z+c) = g(z)$, so that the function $F(z)$ satisfies $F(z) = F(-z-c)$. Hereby if s is a zero of $F(z)$ of order n, the point $-s-c$ is also a zero of $F(z)$ and its order is the same n.

Let a_m be an arbitrary complex number of the sequence $\{a_n\}$, and let s be an arbitrary root of the equation $a(e^z) = a_m$. Because of the manner in which the function $F(z)$ has been constructed, the point s is a zero of $F(z)$ of order p_m. Thereby the point $-s-c$ is also a zero of $F(z)$ of order p_m, and hence the point $-s-c$ is a root of $a(e^z) = a_m$. This means that for every a_m, all the roots of the equation $a(z) = a_m$ are zeros of the rational function $a(z)-a(\omega/z)$, where $\omega = \exp(-c)$. We consequently have $a(z) = a(\omega/z)$, in the whole plane. The same holds for the rational function $b(z)$. This completes the proof of Theorem 2.

5. Proof of Theorem 3

Throughout this section we assume that the function $F(z)$ is prime in entire sense, that is, if $F(z)$ has a factorization $f(g(z))$ with two entire functions $f(z)$ and $g(z)$, then either $f(z)$ or $g(z)$ is linear. Our goal is to show that the function $F(z)$ must be prime under this hypothesis. Suppose that $F(z)$ admits a factorization

$$F(z) = f(g(z)), \tag{5.1}$$

where $f(z)$ and $g(z)$ are meromorphic functions in the whole finite plane. We may further suppose that the left factor $f(z)$ or the right factor $g(z)$ is not entire. If $f(z)$ has a pole at a point s, then $g(z)$ omits the value s. Hence the number of poles of $f(z)$ is at most two. If the right factor $g(z)$ has a pole, then the left factor $f(z)$ must be regular at the point at infinity. Hereby $f(z)$ reduces to a rational function. In particular the case where $f(z)$ is entire and $g(z)$ has a pole is impossible.

We first consider the case where $f(z)$ has exactly two poles. In this case $g(z)$ fails to take two finite values. Hence $g(z)$ must have a pole. Therefore $f(z)$ is rational and has the form

$$f(z) = (z-u)^{-m}(z-v)^{-n}P(z), \tag{5.2}$$

where m and n are positive integers and $P(z)$ is a polynomial of degree at most $m+n$ with $P(u)P(v) \neq 0$. Here we may set $P(z) = \sum_{j=0}^{m+n} \gamma_j z^j$. On the other hand the right factor $g(z)$ can be written as

$$\frac{g(z)-u}{g(z)-v}=\exp(h(z)) \tag{5.3}$$

with a nonconstant entire function $h(z)$. Since $F(z)$ has at least order one and maximal type, and the left factor $f(z)$ is rational in this case, this function $h(z)$ is not linear. Using these representations (5.2) and (5.3), after a simple calculation, we deduce

$$F(z)=f_*(h(z)), \tag{5.4}$$

$$f_*(z)=(u-v)^{-m-n}\exp(-mz)\sum_{j=0}^{m+n}\gamma_j(u-ve^z)^j(1-e^z)^{m+n-j}.$$

Evidently $f_*(z)$ is an entire function, and $\exp(mx)f_*(x)$ converges to the nonzero value $(u-v)^{-m-n}P(u)$ when the real variable x becomes negatively infinite. Hereby $f_*(z)$ is not constant, so it is transcendental. Therefore the above (5.4) is another factorization of $F(z)$, and its left and right factors are both nonlinear entire functions. This is a contradiction.

We next consider the case where $f(z)$ has only one pole and $g(z)$ also has poles. In this case $f(z)$ must be of the form

$$f(z)=(z-u)^{-m}P(z), \tag{5.5}$$

where m is a positive integer and $P(z)$ is a polynomial of degree at most m satisfying $P(u)\neq 0$. Since the right factor $g(z)$ omits the value u,

$$g_*(z)=1/(g(z)-u) \tag{5.6}$$

is a nonconstant entire function. On combining (5.5) and (5.6) we thus obtain the representation

$$\begin{aligned} F(z)&=(g_*(z))^mP(u+1/g_*(z)) \\ &=\sum_{j=0}^{m}\gamma_j(1+ug_*(z))^j(g_*(z))^{m-j}, \end{aligned} \tag{5.7}$$

where $P(z)=\sum_{j=0}^{m}\gamma_jz^j$. Here let us set

$$Q(z)=\sum_{j=0}^{m}\gamma_j(1+uz)^jz^{m-j}.$$

Since $z^{-m}Q(z)$ converges to $P(u)$ when z tends to infinity, this polynomial $Q(z)$ is of degree exactly m. Using the polynomial $Q(z)$ we can rewrite the above expression (5.7) and deduce another factorization $F(z) = Q(g_*(z))$. Evidently two factors $Q(z)$ and $g_*(z)$ are entire functions. It therefore follows from the hypothesis on $F(z)$ that the left factor $Q(z)$ must be a linear function, so that the degree m must be one. Consequently by (5.5), the original left factor $f(z)$ is certainly a linear function. Hereby the factorization (5.1) is trival in this case.

Finally we consider the case where $f(z)$ has exactly one pole and $g(z)$ is an entire function. We may set

$$f(z) = (z - u)^{-m}K(z), \tag{5.8}$$

where $K(z)$ is an entire function with $K(u) \neq 0$ and m is a positive integer. The right factor $g(z)$ omits the value u, and hence with some nonconstant entire function $h(z)$,

$$g(z) = u + \exp(h(z)). \tag{5.9}$$

It therefore follows from (5.8) and (5.9) that

$$F(z) = f_*(h(z)), \qquad f_*(z) = \exp(-mz)K(u+e^z). \tag{5.10}$$

Since $\exp(mz)f_*(z)$ converges to $K(u)$ as e^z tends to zero, the function $f_*(z)$ is transcendental entire. Hereby with the help of the hypothesis on the function $F(z)$, the right factor $h(z)$ must be linear. We therefore have from (5.10) that

$$F(z) = \exp(-m\alpha z - m\beta)K(u+\exp(\alpha z + \beta)), \tag{5.11}$$

where α and β are constants with $\alpha \neq 0$. Evidently $F(z+2\pi i/\alpha) = F(z)$, so that the nonzero constant α is real. For definiteness we may suppose that α is positive. Then by means of Lemma A there exists a sequence $\{t_n\}$ such that $F(t_n) = 0$ for any n and the sequence $\{\exp(\alpha t_n + \beta)\}$ converges to 0 as n goes to infinity. This means from (5.11) that the entire function $K(z)$ vanishes at all the points $u+\exp(\alpha t_n + \beta)$, so that $K(u) = 0$ by continuity. This is a contradiction. Hereby this case is impossible. Consequently the factorization (5.1) is always trivial, so that $F(z)$ is a prime entire function. This completes the proof of Theorem 3.

6. Examples

In this final section we shall present several periodic entire functions which are prime. Before proceeding we want to prove the following final lemma for completeness.

Lemma D. *Let $\{p_n\}$ and $\{q_n\}$ be arbitrary sequences of distinct prime integers such that $p_n \geqq 3$, $q_n \geqq 3$ and $p_m \neq q_n$ for any m and n. Let $a(z)$ and $b(z)$ be arbitrary nonconstant rational functions which are regular in the annulus $0<|z|<+\infty$. Then for any two positive numbers α and β, there exist two sequences $\{a_n\}$ and $\{b_n\}$ of distinct complex numbers, tending to infinity, such that $a_n \neq 0$, $b_n \neq 0$, $a_m \neq b_n$ for any m and n, and the series*

$$(6.1) \qquad \sum_{n\geqq 1} p_n(\log|a_n|)^{-\alpha}, \qquad \sum_{n\geqq 1} q_n(\log|b_n|)^{-\beta}$$

are both convergent, and all the roots of the equations $a(z) = a_n$ and $b(z) = b_n$ are simple only, and the equations $a(z) = a_m$ and $b(z) = b_n$ have no common solutions for any m and n.

Proof of Lemma D. First of all we define the set of points

$$\mathbf{S} = \{a(z) : a'(z) = 0\} \cup \{b(z) : b'(z) = 0\} .$$

Obviously this set $\mathbf{S}$ consists of a finite number of points. We take a point a_1 such that a_1 is not in $\mathbf{S}$, $|a_1| \geqq 1$, and $(\log|a_1|)^\alpha \geqq p_1$. We next consider the set defined with

$$\mathbf{A}_1 = \mathbf{S} \cup \{a_1\} \cup \{b(z) : a(z) = a_1\} .$$

This set $\mathbf{A}_1$ is also a finite set. We take a point b_1 such that b_1 is not in $\mathbf{A}_1$, $|b_1| \geqq 1$ and $(\log|b_1|)^\beta \geqq q_1$. Similarly we set

$$\mathbf{B}_1 = \mathbf{A}_1 \cup \{b_1\} \cup \{a(z) : b(z) = b_1\} ,$$

and take a point a_2 of the complement of $\mathbf{B}_1$ satisfying $|a_2| \geqq 2$ and

$(\log|a_2|)^{\alpha} \geqq 4p_2$. Using this point a_2 we further define the finite set

$$\mathbf{A}_2 = \mathbf{B}_1 \cup \{a_2\} \cup \{b(z) : a(z) = a_2\} ,$$

and take a point b_2 of the complement of $\mathbf{A}_2$ satisfying $|b_2| \geqq 2$ and $(\log|b_2|)^{\beta} \geqq 4q_2$. We continue this process indefinitely, and obtain successively two sequences of finite sets of points $\{\mathbf{A}_n\}_{n\geqq 1}$, $\{\mathbf{B}_n\}_{n\geqq 1}$ and two sequences of complex numbers $\{a_n\}_{n\geqq 1}$, $\{b_n\}_{n\geqq 1}$ such that for every $n \geqq 1$,

$$|a_n| \geqq n, \qquad (\log|a_n|)^{\alpha} \geqq n^2 p_n, \tag{6.2}$$

$$|b_n| \geqq n, \qquad (\log|b_n|)^{\beta} \geqq n^2 q_n, \tag{6.3}$$

$$\mathbf{A}_n = \mathbf{B}_{n-1} \cup \{a_n\} \cup \{b(z) : a(z) = a_n\} , \tag{6.4}$$

$$\mathbf{B}_n = \mathbf{A}_n \cup \{b_n\} \cup \{a(z) : b(z) = b_n\} , \tag{6.5}$$

and a_n is not in the set $\mathbf{B}_{n-1}$, and b_n is a point of the complement of the set $\mathbf{A}_n$, where we set $\mathbf{B}_0 = \mathbf{S}$ for convenience. By means of (6.2) and (6.3), the sequences $\{a_n\}$ and $\{b_n\}$ both tend to infinity and the two series (6.1) are plainly convergent for the positive numbers α and β, respectively. It is clear from (6.4) and (6.5) that $\mathbf{B}_{n-1} \subset \mathbf{A}_n \subset \mathbf{B}_n$ for all $n \geqq 1$. Hereby all the sets $\mathbf{A}_n$ and $\mathbf{B}_n$ contain the set $\mathbf{S}$. Since the point a_n is not in the set $\mathbf{B}_{n-1}$, a_n is not in the set $\mathbf{S}$ either. It thus follows that the equation $a(z) = a_n$ has no multiple solutions. Similarly the point b_n is not in $\mathbf{A}_n$, so b_n is not in the set $\mathbf{S}$. Hereby all the roots of the equation $b(z) = b_n$ must be simple. We take arbitrary two points a_m and a_n with $m<n$. The point a_n is not in the set $\mathbf{B}_{n-1}$, while a_m is a point of the set $\mathbf{A}_m$ by (6.4). Since $\mathbf{A}_m \subset \mathbf{B}_{n-1}$ by $m<n$, the points a_m and a_n are different. Similarly $b_m \neq b_n$ for any different m and n. We next take arbitrary two points a_m and b_n . Again, a_m and b_n are points of the sets $\mathbf{A}_m - \mathbf{B}_{m-1}$ and $\mathbf{B}_n - \mathbf{A}_n$, respectively. If $m \leqq n$, then $\mathbf{A}_m \subset \mathbf{A}_n$, so that the point b_n is not in the set $\mathbf{A}_m$. Hereby $a_m \neq b_n$, and b_n is not contained in the set $\{b(z) : a(z) = a_m\}$ by means of (6.4). It thus follows that the equations $a(z) = a_m$ and $b(z) = b_n$ have no common roots. If $m>n$, then $\mathbf{B}_n \subset \mathbf{B}_{m-1}$, so that a_m is not in $\mathbf{B}_n$. Hence $a_m \neq b_n$, and a_m is not in the set $\{a(z) : b(z) = b_n\}$ by (6.5). Consequently the equations

$a(z) = a_m$ and $b(z) = b_n$ have no common solutions. This completes the proof of Lemma D.

Example 1. Let $\{p_n\}$ and $\{q_n\}$ be sequences of distinct prime integers satisfying $p_n \geqq 3$, $q_n \geqq 3$, and $p_m \neq q_n$ for any m and n. Let $a(z)$ be an arbitrary nonconstant polynomial. Let us set $b(z) = 1/z$. The rational functions $a(z)$ and $b(z)$ are both regular in $0 < |z| < +\infty$, and $a(z)$ has a pole at the infinity while $b(z)$ has a pole at the origin. With the help of the above Lemma D there are two sequences $\{a_n\}$ and $\{b_n\}$ of distinct complex numbers which satisfy the required properties. Consequently using these four sequences $\{a_n\}$, $\{b_n\}$, $\{p_n\}$ and $\{q_n\}$, and the rational functions $a(z)$ and $b(z)$, we can define the periodic entire function

$$F(z) = A(a(e^z))B(b(e^z)) = A(a(e^z))B(e^{-z}),$$

where $A(z)$ and $B(z)$ are the canonical products

$$A(z) = \Pi_{n\geqq 1}\left(1-\frac{z}{a_n}\right)^{p_n}, \qquad B(z) = \Pi_{n\geqq 1}\left(1-\frac{z}{b_n}\right)^{q_n}. \tag{6.6}$$

If $F(z)$ is not prime, by virtue of Theorem 3, the function $F(z)$ satisfies the hypotheses of Theorem 1 or 2. It therefore follows that the rational functions $a(z)$ and $b(z)$ both have poles at the origin and at the point at infinity. Accordingly the function $F(z)$ is certainly prime.

Example 2. Let $\{p_n\}$ and $\{q_n\}$ be as above. Let $\{a_n\}$ and $\{b_n\}$ be arbitrary sequences of distinct complex numbers, tending to infinity, such that $a_n \neq 0$, $a_n \neq \pm 2$, $b_n \neq 0$, $b_n \neq \pm 2i$ and $a_m \neq b_n$, $(a_m+b_n)(a_m-b_n) \neq 4$ for any m and n, and further the series (6.1) are both convergent for some positive α and β. We set $a(z) = z+1/z$ and $b(z) = z-1/z$. Then the rational functions $a(z)$ and $b(z)$ are regular for $0<|z|<+\infty$, and have poles at the origin and at the point at infinity simultaneously. Since $(a(z))^2-(b(z))^2 = 4$, the equations $a(z) = a_m$ and $b(z) = b_n$ have no common solutions for any pair m and n. Of course, by the conditions $a_n \neq \pm 2$ and $b_n \neq \pm 2i$, all the zeros of $a(z)-a_n$ and $b(z)-b_n$ must be simple. Here we set

$$F(z) = A(a(e^z))B(b(e^z)) = A(e^z+e^{-z})B(e^z-e^{-z}),$$

where $A(z)$ and $B(z)$ are the canonical products defined by (6.6). If this function $F(z)$ is not prime, by virtue of Theorem, we can find two nonconstant polynomials $P(z)$ and $Q(z)$, and a nonconstant entire function $h(z)$ which satisfy

$$a(e^z) = P(h(z)), \qquad b(e^z) = Q(h(z)).$$

It then follows that $(P(z))^2-(Q(z))^2 = 4$. This is a contradiction. Hereby the function $F(z)$ must be prime.

Example 3. Let $a(z)$ and $b(z)$ be the rational functions

$$a(z) = z^4+6z^3-25z-9z^{-1}+z^{-2},$$

$$b(z) = z^2+3z+z^{-1},$$

respectively. Let $P(z)$ and $Q(z)$ be the polynomials

$$P(z) = z^6-15z^3+30, \qquad Q(z) = z^3-3,$$

respectively. Then by a routine computation we can see that

$$a(z^3) = P(z^{-1}+z^2), \qquad b(z^3) = Q(z^{-1}+z^2).$$

It therefore follows that

$$a(e^z) = P(g(z)), \qquad b(e^z) = Q(g(z))$$

with $g(z) = \exp(-z/3)+\exp(2z/3)$.

Using these rational functions $a(z)$ and $b(z)$ we construct an entire function $F(z)$ of the form $A(a(e^z))B(b(e^z))$. Evidently $F(z) = f(g(z))$ with the transcendental entire functions $f(z) = A(P(z))B(Q(z))$ and $g(z)$. Thereby the function $F(z)$ admits nontrivial factorizations, that is, $F(z)$ is not prime. This is an example of Theorem 1.

Example 4. We finally present an example for Theorem 2. Let ω be an arbitrary nonzero complex number, and let $a_*(z)$ and $b_*(z)$ be arbitrary nonconstant polynomials. We set $a(z) = a_*(z)+a_*(\omega/z)$ and

$b(z) = b_*(z) + b_*(\omega/z)$. Then it is clear that $a(z)$ and $b(z)$ are both rational and satisfy $a(z) = a(\omega/z)$ and $b(z) = b(\omega/z)$, so that we have the representations

$$a(z) = \sum_{j=0}^{n} \alpha_j C_j(z + \omega/z), \qquad b(z) = \sum_{j=0}^{n} \beta_j C_j(z + \omega/z),$$

where $C_j(z)$ are the polynomials as in the section 1. Furthermore it is also clear that

$$e^z + \omega e^{-z} = \exp(-\gamma)\{\exp(z+\gamma) + \exp(-z-\gamma)\}$$

$$= \exp(-\gamma) D((z+\gamma)^2),$$

where $D(z)$ denotes the entire function $\exp(z^{1/2}) + \exp(-z^{1/2})$, and γ is a constant with $\omega = \exp(-2\gamma)$. We can therefore take two entire functions $U(z)$ and $V(z)$ which satisfy

$$a(e^z) = U((z+\gamma)^2), \qquad b(e^z) = V((z+\gamma)^2).$$

Consequently an entire function of the form $A(a(e^z))B(b(e^z))$ has certainly a factorization $f(g(z))$ with a transcendental entire $f(z)$ and a quadratic polynomial $g(z)$.

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