

Permutability of Certain Entire Functions

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Two entire functions $f(z)$ and $g(z)$ are said to be permutable if they satisfy the identity $f(g(z)) = g(f(z))$. In the paper [3] we indicated a technique by which permutable functions of some certain entire functions can be determined. Indeed we proved the following facts.

Let $f(z)$ be $z + \exp(\alpha z + \beta)$, where α and β are constants with $\alpha \neq 0$. Let $g(z)$ be a nonconstant entire function of finite order which is permutable with $f(z)$. Then either $g(z) = f(z) + \gamma$, or $g(z) = z + \gamma$, where γ is a constant satisfying $\exp(\alpha \gamma) = 1$.

Let $f(z)$ be $z + \sin(z + c)$ with a constant c . Let $g(z)$ be a transcendental entire function of finite order. If the functions $f(z)$ and $g(z)$ are permutable with each other, then either $g(z) = f(z) + \gamma^$ with a constant γ^* satisfying $\cos \gamma^* = 1$, or $g(z) = \gamma_* - f(z)$ with a constant γ_* satisfying $\cos(\gamma_* + 2c) = 1$.*

The purpose of this article is to investigate the possibility of proving further results by making use of our technique. We shall consider two entire functions of exponential type and show the following theorems.

Theorem 1. *Let $f(z)$ be $z + a + b \exp(\gamma z)$, where a , b and γ are constants with $b \gamma \neq 0$. Let $g(z)$ be a nonconstant entire function of finite order which is permutable with $f(z)$. Then either $g(z) = f(z) + \zeta$, or else $g(z) = z + \zeta$, where ζ is a constant with $\exp(\gamma \zeta) = 1$.*

Theorem 2.1. *Let $f(z)$ be $z+a\exp(\gamma z)+c+b\exp(-\gamma z)$, where a, b, c and γ are nonzero constants. Let $g(z)$ be a transcendental entire function of finite order. Suppose that the functions $f(z)$ and $g(z)$ are permutable with each other. Then $g(z) = f(z) + \zeta$, where ζ is a constant with $\exp(\gamma \zeta) = 1$.*

Theorem 2.2. *Let $f(z)$ be $z+a\exp(\gamma z)+b\exp(-\gamma z)$, where a, b and γ are nonzero constants. Let $g(z)$ be a transcendental entire function of finite order which is permutable with $f(z)$. Then either $g(z) = f(z) + \zeta^*$, where ζ^* is a constant with $\exp(\gamma \zeta^*) = 1$, or else $g(z) = -f(z) + \zeta_*$, where ζ_* is a constant satisfying $a\exp(\gamma \zeta_*) = -b$.*

In what follows we assume acquaintance with the standard terminology and the fundamental concepts of Nevanlinna theory, and we use them without further introduction.

1. Let $f(z)$ be $z+a+b\exp(\gamma z)$, where a, b and γ are constants with $b\gamma \neq 0$. Firstly we shall determine all the entire functions of finite order which are permutable with this function $f(z)$.

The function $f(z)$ is entire of exponential type and satisfies the differential equation

$$\begin{aligned} f'(z) &= 1 + \gamma b \exp(\gamma z) \\ (1.1) \quad &= 1 + \gamma \{f(z) - z - a\}. \end{aligned}$$

Evidently the growth of $f(z)$ is

$$(1.2) \quad T(r, f) \sim T(r, \exp(\gamma z)) \sim |\gamma| r / \pi,$$

where $T(r, f)$ denotes the characteristic function of $f(z)$. Since the order of $f(z)$ is finite, the second main theorem asserts that

$$(1.3) \quad (1+o(1))T(r, f) \leq N(r, w_1, f) + N(r, w_2, f) - N(r, 0, f')$$

for all values of r , where w_1 and w_2 are two arbitrary finite complex numbers. On the other hand it is clear from (1.1) that

$$(1.4) \quad N(r, 0, f') = N(r, -1, \gamma \exp(\gamma z)) \sim |\gamma| r / \pi.$$

It therefore follows from (1.2), (1.3) and (1.4) that $N(r, w, f) \sim T(r, f)$ for every finite complex number w . Furthermore by the relation (1.1) again, $f'(z) = 1 + \gamma(w - z - a)$ holds at every w -point of $f(z)$. This means that for each finite complex number w , all the w -points of $f(z)$ are simple except at most for one point. Accordingly we have

$$(1.5) \quad \bar{N}(r, w, f) \sim N(r, w, f) \sim T(r, f)$$

for all finite complex numbers w .

1.1 Now let $g(z)$ be a nonconstant entire function of finite order which satisfies the identity

$$(1.6) \quad f(g(z)) = g(f(z)).$$

Differentiating this identity (1.6) we obtain

$$(1.7) \quad f'(g(z))g'(z) = g'(f(z))f'(z).$$

Suppose for a moment that the derivative $g'(z)$ has no zeros. Then for any zero w of the derivative $f'(z)$, all the w -points of $g(z)$ must be zeros of the derivate $f'(z)$. Therefore

$$\bar{N}(r, w_1, g) + \bar{N}(r, w_2, g) \leq N(r, 0, f')$$

for values of r , where w_1 and w_2 are two distinct zeros of $f'(z)$. Hence $T(r, g) = O(r)$ by the second main theorem. It thus follows that the derivative $g'(z)$ becomes $\exp(\alpha z + \beta)$ with constants α and β . If $\alpha \neq 0$, then the function $g(z)$ has one finite lacunary value. This is impossible by the property (1.5). Hereby $\alpha = 0$, so that the function $g(z)$ is a linear function. We set $g(z) = Az + B$, where A and B are constants with $A \neq 0$. Then with the help of (1.6), the constants A and B satisfy

$$a + b \exp \{ \gamma (Az + B) \} = Aa + Ab \exp(\gamma z),$$

so that $A = 1$ and $\exp(\gamma B) = 1$.

Hereafter we may suppose that the derivative $g'(z)$ has at least one zero point. Let s be a zero of $g'(z)$. Then by the relation (1.7), all the roots of the equation $f(z) = s$ satisfy

$$(1.8) \quad f'(g(z))g'(z) = g'(s)f'(z) = 0,$$

so that either $f'(g(z)) = 0$ or $g'(z) = 0$.

Lemma A. *Let s be a zero of the derivative $g'(z)$. Then all the roots of the equation $f(f(z)) = s$ must be zeros of the entire function*

$$G_s(z) = g'(z) \{g(z) - A_s\} H_s(z),$$

where A_s is the constant defined by $g(s) - 2a + \{1 + \exp(\gamma a - 1) \} \gamma^{-1}$, and $H_s(z)$ is the entire function defined with

$$\begin{aligned} H_s(z) &= g''(s) \{1 - \gamma a + \gamma s - \gamma f(z)\}^2 (f'(z))^2 \\ &\quad + \gamma \{2 - 2\gamma a + \gamma g(s) - \gamma g(z)\}^2 (g'(z))^2. \end{aligned}$$

Proof of Lemma A. Let t be a root of the equation $f(f(z)) = s$. For the sake of simplicity we set $u = f(t)$. Since $f(f(t)) = f(u) = s$ and $g'(s) = 0$, either $g'(u) = 0$ or $f'(g(u)) = 0$ by (1.8).

We first treat the case $g'(u) = 0$. By means of (1.7),

$$f'(g(t))g'(t) = g'(f(t))f'(t) = g'(u)f'(t) = 0.$$

Hence $f'(g(t)) = 0$ or $g'(t) = 0$. Here we may assume that $f'(g(t)) = 0$. For otherwise, $g'(t) = 0$, so that $G_s(t) = 0$. It thus follows from (1.1) that

$$\begin{aligned} 0 &= f'(g(t)) = 1 + \gamma \{f(g(t)) - g(t) - a\} \\ (1.9) \quad &= 1 + \gamma \{g(u) - g(t) - a\} \\ &= 1 + \gamma b \exp(\gamma g(t)). \end{aligned}$$

On the other hand because of $g(s) = g(f(u)) = f(g(u))$,

$$(1.10) \quad g(s) = f(g(u)) = g(u) + a + b \exp(\gamma g(u)).$$

Hence we obtain from (1.9) and (1.10) that

$$\begin{aligned} \gamma g(s) &= 2\gamma a - 1 + \gamma g(t) + \gamma b \exp(\gamma a - 1 + \gamma g(t)) \\ &= 2\gamma a - 1 + \gamma g(t) - \exp(\gamma a - 1). \end{aligned}$$

Hereby $g(t) = A_s$, so that $G_s(z)$ vanishes at the point t .

Secondly we consider the other case $f'(g(u)) = 0$. Again, by virtue of (1.1), (1.7) and (1.8), we have

$$(1.11) \quad f'(g(t))g'(t) = g'(u)f'(t),$$

$$(1.12) \quad f'(u) = 1 + \gamma \{f(u) - u - a\} = 1 + \gamma \{s - u - a\},$$

$$(1.13) \quad f'(g(t)) = 1 + \gamma \{f(g(t)) - g(t) - a\} = 1 + \gamma \{g(u) - g(t) - a\},$$

$$(1.14) \quad 0 = f'(g(u)) = 1 + \gamma \{f(g(u)) - g(u) - a\} = 1 + \gamma \{g(s) - g(u) - a\}.$$

From (1.13) and (1.14), we deduce

$$(1.15) \quad f'(g(t)) = 2 + \gamma \{g(s) - g(t) - 2a\}.$$

On other hand it is clear from (1.7) that

$$\begin{aligned} f''(g(z))(g'(z))^2 + f'(g(z))g''(z) \\ = g''(f(z))(f'(z))^2 + g'(f(z))f''(z). \end{aligned}$$

Hence at the point u , we have

$$(1.16) \quad f''(g(u))(g'(u))^2 = g''(f(u))(f'(u))^2 = g''(s)(f'(u))^2$$

by the assumption $g'(s) = f'(g(u)) = 0$. Here notice the relation

$$f''(z) = \gamma^2 b \exp(\gamma z) = \gamma \{f'(z) - 1\}.$$

It then follows from the assumption $f'(g(u)) = 0$ that

$$f''(g(u)) = \gamma \{f'(g(u)) - 1\} = -\gamma,$$

so that the above (1.16) becomes

$$(1.17) \quad g''(s)(f'(u))^2 + \gamma (g'(u))^2 = 0.$$

Taking (1.11) and this (1.17) into account we thus have

$$(1.18) \quad g''(s) \{f'(u)f'(t)\}^2 + \gamma \{f'(g(t))g'(t)\}^2 = 0.$$

Substituting these (1.12) and (1.15) into (1.18) we finally have

$$(1.19) \quad \begin{aligned} &g''(s) \{1 - \gamma a + \gamma s - \gamma f(t)\}^2 (f'(t))^2 \\ &+ \gamma \{2 - 2\gamma a + \gamma g(s) - \gamma g(t)\}^2 (g'(t))^2 = 0. \end{aligned}$$

This (1.19) yields $H_s(t) = 0$, so that the function $G_s(z)$ surely vanishes at the point t . This completes the proof of Lemma A.

1.2 By means of Lemma A, we have

$$(1.20) \quad N(r, s, f(f(z))) \leq N(r, 0, G_s(z))$$

for values of r . Evidently, by definition, the order of the auxiliary function $H_s(z)$ is finite, and hence the function $G_s(z)$ is also of finite order. On the other hand the twice iterated function $f(f(z))$ has infinite order and it satisfies the identity

$$f(f(z + \zeta)) - f(f(z)) = \zeta$$

with $\zeta = 2\pi i/\gamma$. Therefore for any finite complex number w , the counting function $N(r, w, f(f(z)))$ has infinite order. Consequently with the help of the inequality (1.20), the entire function $G_s(z)$ vanishes identically. Since the entire function $g(z)$ is not constant, the auxiliary

function $H_s(z)$ must be identically equal to zero. Furthermore the value $g''(s)$ is different from zero.

Lemma B. *Let s be an arbitrary zero of the derivative $g'(z)$. Then $g''(s) \neq 0$ and*

$$g''(s) \{1 - \gamma a + \gamma s - \gamma f(z)\}^2 (f'(z))^2 \\ + \gamma \{2 - 2\gamma a + \gamma g(s) - \gamma g(z)\}^2 (g'(z))^2 = 0$$

holds identically. Furthermore the derivative $f'(z)$ always vanishes at the point s .

Proof of Lemma B. It only remains to prove the last statement. On the contrary suppose that there exists a point v such that $g'(v) = 0$ and $f'(v) \neq 0$. Then it is clear by (1.7) that $f'(g(v)) = g'(f(v)) = 0$. Hence we have $g'(f(v)) = 0$. This means that the point $f(v)$ is also a zero of the derivative $g'(z)$. Therefore by virtue of the above argument,

$$g''(f(v)) \{1 - \gamma a + \gamma f(v) - \gamma f(z)\}^2 (f'(z))^2 \\ + \gamma \{2 - 2\gamma a + \gamma g(f(v)) - \gamma g(z)\}^2 (g'(z))^2 = 0$$

for all values of z . Setting $z = v$ in this identity, we at once have

$$g''(f(v))(1 - \gamma a)^2 (f'(v))^2 = 0.$$

Hereby if $\gamma a \neq 1$, we have a contradiction because neither $g''(f(v))$ nor $f'(v)$ is zero. Consequently the assertion is true unless $\gamma a = 1$.

For the case $\gamma a = 1$, the relation

$$(1.21) \quad g''(v) \{v - f(z)\}^2 (f'(z))^2 = -\gamma \{g(v) - g(z)\}^2 (g'(z))^2$$

holds identically. Clearly the right side of (1.21) is zero at the point v . Hence $f(v) = v$. Furthermore it follows from (1.21) that

$$g''(v)(f'(v))^4 = -\gamma (g'(v))^4.$$

This contradicts the assumption $f'(v) \neq 0$. Accordingly the derivative $f'(z)$ vanishes at each zero of the derivative $g'(z)$. This completes the proof of Lemma B.

1.3 We are now in a position to determine the function $g(z)$. By means of the above Lemma B, the identity

$$\begin{aligned} g''(s) \{1 - \gamma a + \gamma s - \gamma f(z)\}^2 (f'(z))^2 \\ + \gamma \{2 - 2\gamma a + \gamma g(s) - \gamma g(z)\}^2 (g'(z))^2 = 0 \end{aligned}$$

holds, where s is a zero of the derivative $g'(z)$. Hence with a suitable constant A , we obtain the identity

$$\begin{aligned} (1.22) \quad & A \{1 - \gamma a + \gamma s - \gamma f(z)\} f'(z) \\ & + \{2 - 2\gamma a + \gamma g(s) - \gamma g(z)\} g'(z) = 0. \end{aligned}$$

Integrating this equation (1.22), we thus find that

$$\begin{aligned} (1.23) \quad & 2A(1 - \gamma a + \gamma s)f(z) - \gamma A(f(z))^2 \\ & + 2(2 - 2\gamma a + \gamma g(s))g(z) - \gamma (g(z))^2 = B, \end{aligned}$$

where B is a certain constant. Here we must determine this constant B . For this purpose, we take a point w such that $\gamma f(w) = 1 - \gamma a + \gamma s$. Then we have $f'(w) = 2 - 2\gamma a + \gamma s - \gamma w$ by (1.7). Hence we may assume that $f'(w)$ is not zero. Then $g'(w) \neq 0$ by virtue of Lemma B, and hence $\gamma g(w) = 2 - 2\gamma a + \gamma g(s)$ by means of (1.22). Hereby the above (1.23) implies

$$(1.24) \quad A(1 - \gamma a + \gamma s)^2 + (2 - 2\gamma a + \gamma g(s))^2 = \gamma B.$$

It therefore follows from (1.23) and (1.24) that

$$A(\gamma f(z)-1+\gamma a-\gamma s)^2+(\gamma g(z)-2+2\gamma a-\gamma g(s))^2=0.$$

Consequently the entire function $g(z)$ can be written in the form

$$g(z) = A^*f(z) + B^*,$$

where A^* and B^* are constants with $A^* \neq 0$. Taking the identity (1.6) into account, after a simple calculation, we finally have $A^* = 1$ and $\exp(\gamma B^*) = 1$. This completes the proof of Theorem 1.

2. Let $f(z)$ be $z + a\exp(\gamma z) + c + b\exp(-\gamma z)$, where a , b , c and γ are constants with $\gamma ab \neq 0$. In this section we shall determine all the entire functions of finite order which are permutable with the function $f(z)$.

The function $f(z)$ satisfies

$$(2.1) \quad f'(z) = 1 + \gamma \{a\exp(\gamma z) - b\exp(-\gamma z)\},$$

$$f''(z) = \gamma^2 \{a\exp(\gamma z) + b\exp(-\gamma z)\}.$$

Hence it follows that

$$(2.2) \quad \gamma^2 \{f(z) - z - c\}^2 - \{f'(z) - 1\}^2 = 4\gamma^2 ab,$$

$$(2.3) \quad \gamma^2 \{f(z) - z - c\} = f''(z),$$

$$(2.4) \quad \{f''(z)\}^2 - \gamma^2 \{f'(z) - 1\}^2 = 4\gamma^4 ab.$$

For convenience we henceforth set the rational function $R(z) = az + c + bz^{-1}$. Then the function $f(z)$ can be written as

$$(2.5) \quad f(z) = z + R(\exp(\gamma z)).$$

We further set the quadratic polynomial $P(z) = \gamma az^2 + z - \gamma b$. It then follows from (2.1) that

$$(2.6) \quad f'(z) \exp(\gamma z) = P(\exp(\gamma z)).$$

By z_1 and z_2 we denote the roots of the quadratic equation $P(z) = 0$. If $4\gamma^2 ab \neq -1$, then these two roots are distinct. For the case $4\gamma^2 ab = -1$, the equation has one double solution. Hence we may interpret as $z_1 = z_2$ in this case. Of course, z_1 and z_2 are both different from zero in any case.

Lemma C. *The function $f(z)$ satisfies*

$$\bar{N}(r, w, f) \sim N(r, w, f) \sim T(r, f) \sim 2 \mid \gamma \mid r / \pi$$

for all finite complex numbers w . Furthermore for an arbitrary finite complex number w , the counting function $N(r, w, f(f(z)))$ has infinite order.

Proof of Lemma C. Since the order of the rational function $R(z)$ is two, it is clear by the representation (2.5) that

$$T(r, f) \sim 2T(r, \exp(\gamma z)) \sim 2 \mid \gamma \mid r / \pi.$$

It is also clear from (2.6) that

$$(2.7) \quad N(r, 0, f'(z)) = N(r, z_1, \exp(\gamma z)) + N(r, z_2, \exp(\gamma z))$$

for all values of r . With the help of this (2.7) and the second main theorem we thus have $N(r, w, f) \sim T(r, f)$ for every finite number w . Moreover let v be a multiple point of $f(z)$. Then we have

$$\gamma(f(v) - v - c)^2 - 1 = 4\gamma^2 ab$$

by means of (2.2). This means that for any finite w , all the w -points of $f(z)$ are simple except at most for two points. Hereby we obtain the first assertion of Lemma C.

Let ζ be a complex number satisfying $\exp(\gamma \zeta) = 1$. Then it is clear that $f(z + \zeta) = f(z) + \zeta$ for all values of z . Therefore the twice iterated function $f(f(z))$ satisfies the identity

$$f(f(z + \zeta)) - f(f(z)) = \zeta.$$

Hence by the well known Picard-Borel theorem, for any finite complex number w , the counting function $N(r, w, f(f(z)))$ is of the same order as the characteristic function $T(r, f(f(z)))$. Furthermore it is easily checked that the multiplicity at any point of $f(z)$ is at most three, so that that of the iteration $f(f(z))$ is at most nine. Consequently the counting function $N(r, w, f(f(z)))$ is of infinite order for any finite complex number w . The proof of Lemma C is now complete.

2.1 Let $g(z)$ be a nonconstant entire function of finite order which is permutable with the function $f(z)$. As in the previous section we then have the identities

$$(2.8) \quad f(g(z)) = g(f(z)),$$

$$(2.9) \quad f'(g(z))g'(z) = g'(f(z))f'(z).$$

Our next task is to show the following

Lemma D. *Suppose that the derivative $g'(z)$ has at most finitely many zeros. Then the function $g(z)$ reduces to a linear function.*

Proof of Lemma D. Suppose first that the derivative $g'(z)$ has m distinct zeros. We denote these points with $\{v_j : j = 1, 2, \dots, m\}$. Let n be an arbitrary positive integer and let $\{w_k : k = 1, 2, \dots, n\}$ be arbitrarily chosen n distinct zeros of derivative $f'(z)$. It then follows from (2.9) that every solution of the equation $g(z) = w_k$ satisfies $f'(z) = 0$ or $f(z) = v_j$ for some j . Hence we have

$$\sum_{k=1}^n N(r, w_k, g) \leq N(r, 0, f') + \sum_{j=1}^m N(r, v_j, f)$$

for values of r , and hence the second main theorem and the above Lemma C yield

$$(2.10) \quad \{n-1+o(1)\}T(r, g) \leq 2(m+1) \mid \gamma \mid r/\pi$$

as r tends to infinity. Since m is fixed and n can be chosen as large as we please, it thus follows that $T(r, g) = o(r)$. This means that the function $g(z)$ reduces to a polynomial.

Suppose next that the derivative $g'(z)$ has no zeros. It then follows from (2.9) that

$$\sum_{k=1}^n N(r, w_k, g) \leq N(r, 0, f')$$

for values of r , where $\{w_k\}$ are arbitrarily chosen n distinct zeros of the derivative $f'(z)$. Hence the above (2.10) remains valid with $m = 0$. Accordingly the function $g(z)$ also reduces to a polynomial.

Let k denote the degree of the polynomial $g(z)$. Then the order of the composite function $f(g(z))$ is precisely equal to k . On the other hand the composite $g(f(z))$ is still of exponential type. In particular, its order is exactly one. Consequently the degree k must be one, so that $g(z)$ must be linear. Thus Lemma D is proved.

We set $g(z) = Az + B$ with two constants A and B . Then $f(Az + B) = Af(z) + B$ by (2.8). Hence by the definition of $f(z)$, the identity

$$\begin{aligned} & Aa\exp(\gamma z) + Ac + Ab\exp(-\gamma z) \\ (2.11) \quad & = a\exp(\gamma Az + \gamma B) + c + b\exp(-\gamma Az - \gamma B) \end{aligned}$$

holds. Differentiating both sides we have

$$a\exp(\gamma z) - b\exp(-\gamma z) = a\exp(\gamma Az + \gamma B) - b\exp(-\gamma Az - \gamma B),$$

so that

$$\{\exp(\gamma Az + \gamma B) - \exp(\gamma z)\} \{b + a\exp(\gamma z + \gamma Az + \gamma B)\} = 0.$$

Consequently $A = 1$ and $\exp(\gamma B) = 1$, or $A = -1$ and $a\exp(\gamma B) = -b$.

Furthermore we can easily see from (2.11) that the latter case occurs only if $c = 0$, that is, only for the function $f(z) = z + a \exp(\gamma z) + b \exp(-\gamma z)$. Accordingly for the function $f(z) = z + a \exp(\gamma z) + c + b \exp(-\gamma z)$ with a nonzero constant c , a linear function which is permutable with $f(z)$ is necessarily of the form $z + \zeta$, where ζ is a constant satisfying $\exp(\gamma \zeta) = 1$. For the function $f(z) = z + a \exp(\gamma z) + b \exp(-\gamma z)$, a linear function which is permutable with $f(z)$ is either of the form $z + \zeta^*$ with a constant ζ^* satisfying $\exp(\gamma \zeta^*) = 1$, or of the form $\zeta^* - z$ with a constant ζ^* satisfying $a \exp(\gamma \zeta^*) = -b$.

2.2 Hereafter we may assume that the derivative $g'(z)$ has infinitely many zeros. Let s be a zero of $g'(z)$. Then with the help of (2.9),

$$(2.12) \quad f'(g(z))g'(z) = g'(s)f'(z) = 0$$

for all the roots of the equation $f(z) = s$. Let u and t be any points such that $f(u) = s$ and $f(t) = u$. Evidently, $f'(g(u)) = 0$ or $g'(u) = 0$ by (2.12).

Suppose that $g'(u) = 0$. Then this point u is also zero of the derivative $g'(z)$. Hence $f'(g(t)) = 0$ or $g'(t) = 0$ by (2.9) again. If the former case occurs, then by (2.6), the quadratic polynomial $P(z)$ vanishes at the point $\exp(\gamma g(t))$. Hence $\exp(\gamma g(t)) = z_j$ for $j = 1$ or 2 , so that

$$g(u) = f(g(t)) = g(t) + R(z_j),$$

by the representation (2.5). Consequently by making use of $g(s) = f(g(u))$, we obtain

$$g(t) = g(s) - R(z_j) - R(\zeta_j),$$

where $\zeta_j = z_j \exp(\gamma R(z_j))$. Accordingly if $g'(u) = 0$, then all the u -points of $f(z)$ are also zeros of the entire function

$$(2.13) \quad G(z, s) = g'(z) \{g(z) - g(s) + A_1\} \{g(z) - g(s) + A_2\},$$

where $A_j = R(z_j) + R(\zeta_j)$ ($j = 1, 2$).

We next consider the case $g'(u) \neq 0$. In this case we have necessarily $f'(g(u)) = 0$. Differentiating (2.9) once again we thus have

$$(2.14) \quad g''(s)(f'(u))^2 = f''(g(u))(g'(u))^2.$$

On the other hand with the help of (2.6), $\exp(\gamma g(u)) = z_1$ or $\exp(\gamma g(u)) = z_2$, where z_1 and z_2 are the zeros of the quadratic polynomial $P(z)$, and hence

$$(2.15) \quad g(s) = f(g(u)) = g(u) + R(z_j)$$

for $j = 1$ or 2 . Furthermore with the help of (2.3),

$$(2.16) \quad f''(g(u)) = \gamma^2 \{f(g(u)) - g(u) - c\} = \gamma^2 \{g(s) - g(u) - c\}.$$

We consequently obtain from (2.14), (2.15) and (2.16) that

$$(2.17) \quad g''(s)(f'(u))^2 = \gamma^2 \{R(z_j) - c\}(g'(u))^2$$

for $j = 1$ or 2 . By an elementary computation we can easily see that

$$\gamma^2 \{R(z_j) - c\}^2 = 4\gamma^2 ab + 1$$

for both the zeros z_1 and z_2 . Hence if $4\gamma^2 ab$ is not equal to -1 , then the twice derivative $g''(s)$ is different from zero by (2.17) and the assumption $g'(u) \neq 0$. For the case $4\gamma^2 ab = -1$ we further deduce

$$(2.18) \quad g^{(3)}(s)(f'(u))^3 = -\gamma^2 (g'(u))^3.$$

In fact with the help of (2.4), $f''(g(u)) = 0$ because of $f'(g(u)) = 0$ and $4\gamma^2 ab + 1 = 0$. By differentiating (2.3) and setting $z = g(u)$, we at once have $f^{(3)}(g(u)) = -\gamma^2$. On the other hand by differentiation we deduce

$$g^{(3)}(s)(f'(u))^3 + 3g''(s)f'(u)f''(u) = f^{(3)}(g(u))(g'(u))^3.$$

Since $g''(s)(f'(u))^2 = 0$ in this case, we obtain (2.18) immediately. By the assumption $g'(u) \neq 0$, the quantity $g^{(3)}(s)$ does not vanish.

Lemma E. *Let $g(z)$ be a transcendental entire function of finite order which is permutable with $f(z)$. Then the derivative $g'(z)$ has infinitely many zeros.*

Let s be an arbitrary zero of the derivative $g'(z)$. Then there exists a point such that $f(z) = s$ and $g'(z) \neq 0$. Furthermore if $4\gamma^2 ab + 1 = 0$, the quantity $g^{(3)}(s)$ is different from zero and

$$g(z) = g(s) - c, \quad g^{(3)}(s)(f'(z))^3 = -\gamma^2 (g'(z))^3$$

hold for all the points such that $f(z) = s$ and $g'(z) \neq 0$. If $4\gamma^2 ab + 1$ is different from 0, then $g''(s)$ does not vanish and for $j = 1$ or $j = 2$

$$g(z) = g(s) - R(z_j), \quad g''(s)(f'(z))^2 = \gamma^2 \{R(z_j) - c\} (g'(z))^2$$

hold simultaneously at each point satisfying $f(z) = s$ and $g'(z) \neq 0$, where z_1 and z_2 are the zeros of the quadratic polynomial $P(z)$.

Proof of Lemma E. Let s be an arbitrary zero of $g'(z)$. Suppose that all the s -point of $f(z)$ are also zeros of $g'(z)$. Then by what we have seen just above, the entire function $G(z, s)$ defined with (2.13) vanishes at all the points satisfying $f(f(z)) = s$. This yields

$$N(r, s, f(f(z))) \leq N(r, 0, G(z, s))$$

for values of r . The order of $G(z, s)$ is finite, so that the right side of this inequality is of finite order. On the other hand the left side is of infinite order by Lemma C. Consequently the entire function $G(z, s)$ vanishes identically. This is a contradiction because $g(z)$ is transcendental. Consequently for any zero point s of $g'(z)$, there is at least one point which satisfies $f(z) = s$ and $g'(z) \neq 0$. This completes the proof of Lemma E.

Again let s be a zero of $g'(z)$ and let u be any point satisfying $f(u) = s$ and $g'(u) \neq 0$. Furthermore let t be an arbitrary point such that $f(t) = u$. We recall the identity (2.2). Then by setting $z = u$ in (2.2), we have

$$\gamma^2 \{s - u - c\}^2 - \{f'(u) - 1\}^2 = 4\gamma^2 ab.$$

It thus follows from Lemma E that

$$(2.19) \quad \{Ag'(u)-1\}^2 = \gamma^2 \{s-u-c\}^2 - 4\gamma^2 ab,$$

where A is a nonzero constant with $A^3 = -\gamma^2/g^{(3)}(s)$ if $4\gamma^2 ab+1=0$, and with $A^2 = \gamma^2(R(z_j)-c)/g''(s)$ if $4\gamma^2 ab+1 \neq 0$. We multiply both sides of this (2.19) by $(f'(t))^2$. Then with the help of (2.9) we easily have

$$(2.20) \quad \{Af'(g(t))g'(t)-f'(t)\}^2 = (\gamma f'(t))^2 \{(s-f(t)-c)^2 - 4ab\}.$$

Recall the identity (2.2) again. This time by setting $z = g(t)$ we deduce

$$\gamma^2 \{f(g(t))-g(t)-c\}^2 - \{f'(g(t))-1\}^2 = 4\gamma^2 ab,$$

so that by means of (2.15),

$$\{f'(g(t))-1\}^2 = \gamma^2 \{g(s)-R(z_j)-g(t)-c\}^2 - 4\gamma^2 ab.$$

Here we should remark that the quantity $R(z_j)$ is the same one which appears in the definition of the constant A . In particular $R(z_j) = c$ for the case $4\gamma^2 ab+1=0$. Multiplying both sides by $(Ag'(t))^2$, we then obtain

$$(2.21) \quad \begin{aligned} & \{Af'(g(t))g'(t)-Ag'(t)\}^2 \\ &= \{\gamma Ag'(t)\}^2 \{(g(s)-R(z_j)-g(t)-c)^2 - 4ab\}. \end{aligned}$$

We now consider the difference of (2.20) and (2.21). For convenience we set

$$(2.22) \quad \begin{aligned} E(t, s) &= (\gamma f'(t))^2 \{(s-f(t)-c)^2 - 4ab\} \\ &\quad - (\gamma Ag'(t))^2 \{(g(s)-R(z_j)-g(t)-c)^2 - 4ab\}. \end{aligned}$$

Then it follows from (2.20) and (2.21) that

$$\begin{aligned} E(t, s) &= \{Af'(g(t))g'(t) - f'(t)\}^2 - \{Af'(g(t))g'(t) - Ag'(t)\}^2 \\ &= \{2Af'(g(t))g'(t) - f'(t) - Ag'(t)\} \{Ag'(t) - f'(t)\}, \end{aligned}$$

so that we obtain

$$\begin{aligned} (2.23) \quad & 2 \{Af'(g(t))g'(t) - f'(t)\} \{Ag'(t) - f'(t)\} \\ &= E(t, s) + \{Ag'(t) - f'(t)\}^2. \end{aligned}$$

Substituting this (2.23) into (2.20), we consequently deduce

$$\begin{aligned} (2.24) \quad & \{E(t, s) + (Ag'(t) - f'(t))^2\}^2 \\ &= (2 \gamma f'(t))^2 \{Ag'(t) - f'(t)\}^2 \{(s - f(t) - c)^2 - 4ab\}. \end{aligned}$$

We are now in a position to prove the following

Lemma F. *Let $g(z)$ be a transcendental entire function of finite order which is permutable with $f(z)$. Let s be an arbitrary zero point of $g'(z)$. Then the identity*

$$\begin{aligned} & \{E(z, s) + (Ag'(z) - f'(z))^2\}^2 \\ &= (2 \gamma f'(z))^2 \{Ag'(z) - f'(z)\}^2 \{(s - f(z) - c)^2 - 4ab\} \end{aligned}$$

holds, where $E(z, s)$ is entire of z given by

$$\begin{aligned} E(z, s) &= (\gamma f'(z))^2 \{(s - f(z) - c)^2 - 4ab\} \\ &\quad - (\gamma Ag'(z))^2 \{(g(s) - g(z) - B)^2 - 4ab\}, \end{aligned}$$

and A is a nonzero constant, and $B = R(z_1) + c$ or $B = R(z_2) + c$.

Proof of Lemma F. We distinguish two cases. We first consider the

case $4\gamma^2 ab + 1 = 0$. Let s be an arbitrary zero of $g'(z)$. By A_1 , A_2 and A_3 we denote the three roots of the equation $g^{(3)}(s)z^3 + \gamma^2 = 0$. We now set the three functions

$$E_j(z, s) = (\gamma f'(z))^2 \{(s - f(z) - c)^2 - 4ab\} \\ - (\gamma A_j g'(z))^2 \{(g(s) - g(z) - 2c)^2 - 4ab\} \quad (j = 1, 2, 3).$$

Evidently these functions of z are entire and of finite order. By using these functions we further set

$$H_j(z, s) = \{E_j(z, s) + (A_j g'(z) - f'(z))^2\}^2 \\ - (2\gamma f'(z))^2 \{A_j g'(z) - f'(z)\}^2 \{(s - f(z) - c)^2 - 4ab\}$$

for $j = 1, 2, 3$. These three functions of z are also entire and of finite order. Let u be an arbitrary point satisfying $f(u) = s$ and $g(u) \neq 0$. Then with the help of Lemma E, $g(u) = g(s) - c$ and $f'(u) = A_j g'(u)$ for some j . Hence (2.19) holds with $A = A_j$. Furthermore it is clear that the relations (2.20) and (2.21) with $A = A_j$ hold good for all the points t satisfying $f(t) = u$. Since $R(z_j) = c$ in this case, the quantity $E(t, s)$ of (2.22) is equal to the value $E_j(t, s)$ for this index j . Consequently by virtue of (2.24), we have $H_j(t, s) = 0$ for all the points t such that $f(t) = u$. Accordingly for every point u such that $f(u) = s$ and $g'(u) \neq 0$, one of the three functions $H_j(z, s)$ vanishes at all the u -points of $f(z)$. For a point u such that $f(u) = s$ and $g'(u) = 0$, the entire function $G(z, s)$ must vanish at every u -point of $f(z)$. It therefore follows that the product $G(z, s)$ and the three $H_j(z, s)$ always vanishes at all the s -points of the iterated $f(f(z))$. By this fact and by Lemma C, we finally conclude that one of the four functions $G(z, s)$ and $H_j(z, s)$ ($j = 1, 2, 3$) vanishes identically. Since $g(z)$ is transcendental, $G(z, s)$ is not constant zero. Hereby one of the three $H_j(z, s)$ reduces to the constant zero. This is precisely what we have to prove.

Next we consider the case where $4\gamma^2 ab + 1$ is different from zero. Again let s be an arbitrary zero of $g'(z)$. For this case by A_{1j} and A_{2j} we denote the square roots of the values $\gamma^2(R(z_j) - c)/g''(s)$ ($j = 1, 2$). Using these four quantities we set the functions

$$\begin{aligned}
 E_j(z, s) &= (\gamma f'(z))^2 \{(s - f(z) - c)^2 - 4ab\} \\
 &\quad - (\gamma A_{ij}g'(z))^2 \{(g(s) - g(z) - R(z_j) - c)^2 - 4ab\}, \\
 H_{ij}(z, s) &= \{E_j(z, s) + (A_{ij}g'(z) - f'(z))^2\}^2 \\
 &\quad - (2\gamma f'(z))^2 \{A_{ij}g'(z) - f'(z)\}^2 \{(s - f(z) - c)^2 - 4ab\}
 \end{aligned}$$

for $i, j = 1, 2$. Since $A_{1j} = -A_{2j}$, the functions $E_j(z, s)$ are independent of the first index i . As before these functions of z are entire and of finite order. Let u be an arbitrary point satisfying $f(u) = s$ and $g(u) \neq 0$. Then with the help of Lemma E again, $g(u) = g(s) - R(z_j)$ and $f'(u) = A_{ij}g'(u)$ for some i and j . It therefore follows from (2.20)-(2.24) and the definitions of $E_j(z, s)$ and $H_{ij}(z, s)$ that for this pair i and j , the function $H_{ij}(z, s)$ vanishes at all the u -points of $f(z)$. Repeating exactly the same argument used in the previous case, we consequently see that all the s -points of $f(f(z))$ are also zeros of the product $G(z, s)$ and the four functions $H_{ij}(z, s)$ ($i, j = 1, 2$). Accordingly one of the four functions $H_{ij}(z, s)$ must vanish identically. This completes the proof of Lemma F.

2.3 The next lemma is an immediate consequence of the above Lemma F.

Lemma G. *Let $g(z)$ be a transcendental entire function of finite order. Suppose that $f(z)$ and $g(z)$ are permutable with each other. Then the derivatives $f'(z)$ and $g'(z)$ have the same zeros.*

Proof of Lemma G. Let s be an arbitrary zero of $g'(z)$. Then we have the identity

$$\begin{aligned}
 &\{E(z, s) + (Ag'(z) - f'(z))^2\}^2 \\
 (2.25) \quad &= (2\gamma f'(z))^2 \{Ag'(z) - f'(z)\}^2 \{(s - f(z) - c)^2 - 4ab\},
 \end{aligned}$$

where $E(z, s)$ is entire of z given by

$$E(z, s) = (\gamma f'(z))^2 \{(s - f(z) - c)^2 - 4ab\} \\ - (\gamma Ag'(z))^2 \{(g(s) - g(z) - B)^2 - 4ab\},$$

and A is a nonzero constant, and $B = R(z_1) + c$ or $B = R(z_2) + c$. Let w be another zero of $g'(z)$. Then by setting $z = w$ in the identity (2.25) we have

$$\{E(w, s) + (f'(w))^2\}^2 = (2\gamma f'(w))^2 \{f'(w)\}^2 \{(s - f(w) - c)^2 - 4ab\}, \\ E(w, s) = (\gamma f'(w))^2 \{(s - f(w) - c)^2 - 4ab\}.$$

It therefore follows that $f'(w) = 0$ or $\gamma^2 \{s - f(w) - c\}^2 = 1 + 4\gamma^2 ab$. Consequently if there is a zero w of $g'(z)$ such that $f'(w) \neq 0$, then any other zero s of $g'(z)$ must satisfy $\gamma^2 \{s - f(w) - c\}^2 = 1 + 4\gamma^2 ab$. However the derivative $g'(z)$ has infinitely many zeros. This is a contradiction. Accordingly all the zeros of $g'(z)$ are also zeros of $f'(z)$.

Now let s be an arbitrary zero of $g'(z)$ again, and let v be any zero of $f'(z)$. We set $z = v$ in the identity (2.25). It then follows that

$$(Ag'(v))^2 = (\gamma Ag'(v))^2 \{(g(s) - g(v) - B)^2 - 4ab\}.$$

Hereby either $g'(v) = 0$ or

$$\gamma^2 (g(s) - g(v) - B)^2 = 1 + 4\gamma^2 ab,$$

where $B = R(z_1) + c$ or $B = R(z_2) + c$. We now suppose that there exists a point v satisfying $f'(v) = 0$ and $g'(v) \neq 0$. Then at each zero of $g'(z)$

$$(2.26) \quad \gamma^2 (g(z) - g(v) - R(z_j) - c)^2 = 1 + 4\gamma^2 ab$$

holds for $j = 1$ or 2 . Let t be a point satisfying $\exp(\gamma g(t)) = z_i$ for $i = 1$ or 2 . Then it is clear from (2.5) that

$$(2.27) \quad g(f(t)) = f(g(t)) = g(t) + R(z_i).$$

On the other hand it follows from (2.6) that $f'(g(t)) = 0$, so that $f'(t) = 0$ or $f(t)$ is a zero of $g'(z)$ by (2.9). Hence with the help of (2.26), either $f'(t) = 0$, or else

$$(2.28) \quad \gamma^2(g(f(t)) - g(v) - R(z_j) - c)^2 = 1 + 4\gamma^2 ab$$

for $j = 1$ or 2 . Combining (2.27) with this (2.28) we consequently have

$$\gamma^2(g(t) - g(v) + R(z_i) - R(z_j) - c)^2 = 1 + 4\gamma^2 ab.$$

This means that

$$\begin{aligned} & N(r, z_1, \exp(\gamma g(z))) + N(r, z_2, \exp(\gamma g(z))) \\ & \leq N(r, 0, f'(z)) + \sum_{k=1}^6 N(r, w_k, g) \end{aligned}$$

for values of r , where w_k ($k = 1, 2, \dots, 6$) are certain complex numbers. Since the roots z_1 and z_2 are not zero and the function $g(z)$ is transcendental, the left side of this inequality has infinite order. However the right side is clearly of finite order. This is a contradiction again. Consequently all the zeros of $f'(z)$ are also zeros of $g'(z)$.

For the case where $4\gamma^2 ab + 1 \neq 0$ the derivative $f'(z)$ has simple zeros only, and the derivative $g'(z)$ has also simply zeros only by means of Lemma E. Accordingly we complete the proof for this case.

For the case where $4\gamma^2 ab + 1 = 0$ the derivative $f'(z)$ has double zeros only. Let s be a zero of the derivative $g'(z)$. Then with the help of Lemma E, there exists a point u such that $f(u) = s$ and $g'(u) \neq 0$. Because of $f'(g(u)) = 0$, $g''(s)(f'(u))^2 = 0$. On the other hand by what we have just proved, $f'(u)$ must be different from zero. It thus follows that $g''(s) = 0$. Thereby with the help of Lemma E again, the point s is a double zero of the derivative $g'(z)$. Consequently the derivatives $f'(z)$ and $g'(z)$ have exactly the same zeros. The proof of Lemma G is now complete.

2.4 By virtue of this Lemma G, the quotient $g'(z)/f'(z)$ has no poles and no zeros. Since this quotient is of finite order, we can write

$$(2.29) \quad g'(z) = f'(z)\exp(H(z)),$$

where $H(z)$ is a suitable polynomial. From now on we want to show that this polynomial $H(z)$ is constant, that is, $g(z) = Af(z) + B$ with constants A and B .

We first consider the case where the degree of $H(z)$ is more than one. Let us set

$$H(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

with $a_n \neq 0$. Let α denote an argument of a_n , and let δ be an arbitrary real positive number with $\delta < \pi/2n$. Then it is possible to find a number R such that

$$|H(z) - a_n z^n| < \varepsilon |z|^n$$

for all values of z with $|z| > R$, where $2\varepsilon = |a_n| \cos n\delta$. It therefore follows that

$$(2.30) \quad \operatorname{Re}\{H(z)\} \geq \varepsilon |z|^n$$

for values of z with $|z| > R$ and $|\arg z + \alpha/n| < \delta$, and it also follows that

$$(2.31) \quad \operatorname{Re}\{H(z)\} \leq -\varepsilon |z|^n$$

for values of z with $|z| > R$ and $|\arg z + (\alpha + \pi)/n| < \delta$. By making use of the relation (2.29) and the inequality (2.30),

$$(2.32) \quad \begin{aligned} \log |g'(z)| &= \operatorname{Re}\{H(z)\} + \log |f'(z)| \\ &\geq \varepsilon |z|^n + \log |f'(z)| \end{aligned}$$

for values of z with $|z| > R$ and $|\arg z + \alpha/n| < \delta$.

Let η be a real number such that $|\eta + \alpha/n| < \delta$ and that $\eta + \arg \gamma$ is not a multiple of $\pi/2$. Then we can see that when z tends to infinity along the ray $L: \arg z = \eta$, either

$$f'(z)\exp(-\gamma z) \rightarrow \gamma a \quad \text{or} \quad f'(z)\exp(\gamma z) \rightarrow -\gamma b$$

holds according to whether $\cos(\eta + \arg \gamma)$ is positive or negative. Thereby from (2.32), the function $g'(z)$ approaches infinity as z goes to infinity along the ray L .

On the other hand it follows from (2.29) and (2.31) that

$$\begin{aligned} \log |g'(z)| &= \operatorname{Re} \{H(z)\} + \log |f'(z)| \\ (2.33) \quad &\leq -\varepsilon |z|^n + \log |f'(z)| \end{aligned}$$

for values of z with $|z| > R$ and $|\arg z + (\alpha + \pi)/n| < \delta$. Since $\log |f'(z)| = O(|z|)$ as z tends to infinity, we can thus see from (2.33) that the function $g(z)$ converges to some finite value when z tends to infinity from the inside of the angular region $|\arg z + (\alpha + \pi)/n| < \delta$. In particular $g(z)$ is bounded in this angular region. Here it is clear from (2.9) and (2.29) that

$$g'(f(z)) = f'(g(z))\exp(H(z)).$$

Since the composite $f'(g(z))$ must be bounded in $|\arg z + (\alpha + \pi)/n| < \delta$, by virtue of (2.31), we consequently have

$$(2.34) \quad \lim_{z \rightarrow \infty} g'(f(z)) = 0$$

uniformly for $|\arg z + (\alpha + \pi)/n| < \delta$.

Let θ be a real number such that $|\theta + (\alpha + \pi)/n| < \delta$ and that $\theta + \arg \gamma$ is not a multiple of $\pi/2$. Then as before we can see that according to whether $\cos(\theta + \arg \gamma)$ is positive or negative,

$$f(z)\exp(-\gamma z) \rightarrow a \quad \text{or} \quad f(z)\exp(\gamma z) \rightarrow b$$

as z tends to infinity along the ray $\arg z = \theta$. Hereby $f(z)$ approaches infinity, and the argument of $f(z)$ becomes positively or negatively infinite along this ray $\arg z = \theta$. Consequently the curve $C = \{f(z) : \arg z = \theta\}$ is asymptotic and it winds around the point at infinity infinitely many times. Of course, the above (2.34) means that the

derivative $g'(z)$ has the value zero as an asymptotic value along this curve C . Furthermore we can easily see that the curve C intersects the ray L in any neighbourhood at the point at infinity. However this is a contradiction because the derivative $g'(z)$ converges to infinity along the ray L . Accordingly the degree of the polynomial $H(z)$ is zero or one.

We next consider the case in which $H(z)$ is nonconstant and linear. We set $H(z) = \alpha z + \beta$. Again let s be a zero of the derivative $g'(z)$, and let u be a root of the equation $f(z) = s$. If $g'(u) = 0$, then $f'(u) = 0$ by means of Lemma G. Hence $\gamma^2(s-u-c)^2 - 4\gamma^2 ab = 1$ by (2.2). Consequently all the roots of the equation $f(z) = s$ satisfy $g'(z) \neq 0$, and hence $f'(g(z)) = 0$ except at most two roots. Let $\{v_j\}$ be the sequence of all the points such that $f(z) = s$ and $g'(z) \neq 0$. Then it follows that

$$g''(s)(f'(v_j))^2 = f''(g(v_j))(g'(v_j))^2,$$

so that by virtue of (2.29), $g''(s) = f''(g(v_j))\exp(2H(v_j))$. Furthermore with the help of (2.4), $\{f''(g(v_j))\}^2 = \gamma^2 + 4\gamma^2 ab$. Accordingly $\exp(4H(v_j))$ are constant, that is, $\exp(4\alpha v_j)$ are constant for all j . It therefore follows that the real parts of αv_j are constant. On the other hand by an asymptotic behavior of the function $f(z)$, for any positive ε , all but a finite number of v_j satisfy $|\arg \gamma v_j - \pi/2| < \varepsilon$ or $|\arg \gamma v_j - 3\pi/2| < \varepsilon$. Since $\{v_j\}$ converges to the point at infinity, it thus follows that the quotient α/γ must be real. However this is clearly impossible because the sequence $\{\exp(\gamma v_j)\}$ is not bounded.

Consequently the quotient $g'(z)/f'(z)$ must be constant. Hence the function $g(z)$ can be written as $g(z) = Af(z) + B$ with two constants A and B . It only remains to determine these constants. Let us recall the identity (2.8). Then $Af(f(z)) + B = f(Af(z) + B)$. Hereby the constants A and B satisfy

$$\begin{aligned} (2.35) \quad & a\exp(\gamma Af(z) + \gamma B) + c + b\exp(-\gamma Af(z) - \gamma B) \\ & = Aa\exp(\gamma f(z)) + Ac + Ab\exp(-\gamma f(z)). \end{aligned}$$

On differentiating both sides we have the identity

$$\begin{aligned} & a \exp(\gamma A f(z) + \gamma B) - b \exp(-\gamma A f(z) - \gamma B) \\ &= a \exp(\gamma f(z)) - b \exp(-\gamma f(z)). \end{aligned}$$

It therefore follows that either $A = 1$ and $\exp(\gamma B) = 1$, or $A = -1$ and $a \exp(\gamma B) = -b$. Assume now that the latter case occurs. Then the above (2.35) yields that $c = -c$. Hence the constant c must be zero. Accordingly the latter case occurs only for the function $f(z) = z + a \exp(\gamma z) + b \exp(-\gamma z)$. We have now completed the proofs of Theorems 2.1. and 2.2.

References

- [1] F. Gross, Factorization of meromorphic functions, Math. Research Center, Washington D. C., 1972.
- [2] W. K. Hayman, Meromorphic functions, Oxford Univ. Press, 1964.
- [3] T. Kobayashi, Permutability and unique factorizability of certain entire functions, Kodai Math. J., 3 (1980), 8-25.