

中央に隔壁のある矩形内の2次元 ポテンシャル循環流

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Two-Dimensional Potential Circulating Flow in the
Rectangle with the Separating plane Wall.

by Yukio Ozaki

Résumé : 矩形の中央に、その一辺に平行な隔壁のある場合の矩形内の二次元ポテンシャル循環流の複素速度ポテンシャル、流れの関数および速度ポテンシャルを与える式を等角写像の方法によって求めることができた。これらの流れの関数および速度ポテンシャルを与える式を図式的に解くことにより、矩形内の二次元循環流の速度分布および流線が与えられる。

Résumé; The complex velocity potential of the two-dimensional circulating flow in the rectangle is obtained by means of the method of conformal representation, when there is a separating plane wall in it. The stream function and the velocity potential are also obtained from this complex velocity potential, and the numerical values of them are able to be obtained by means of the graphical method.

§ 1 Introduction

Many studies are published on the two-dimensional potential flow around the corner with some value of angle¹⁻¹¹⁾. The study on the U-turn flow at the end of the rectangle is published by the author.⁷⁾

But few studies are published on the circulating flow in the rectangle. In this paper, the author intends to study the two-dimensional circulating potential flow in the rectangle with a plane central wall parallel with its side. The author also intends to investigate the stream function and the velocity potential, from which the stream lines and the velocity distribution will be obtained, and these will give the useful knowledge for the design of the water channel.

§ 2 Theoretical.

The flow in the rectangle is studied which has a separating wall in the center as shown in Fig. 1. If the z -plane shown in Fig. 2. is used, the complex velocity potential w will be the elliptic integral as follows. :

$$w = -i \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2)}} .$$

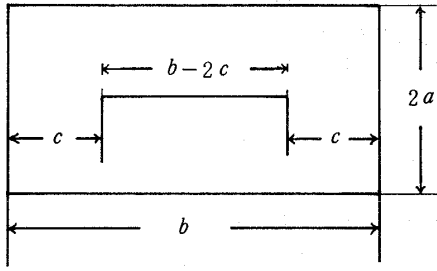


Fig. 1 t-plane

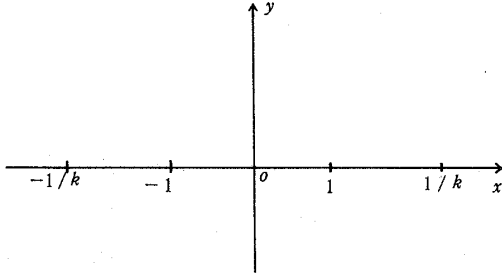


Fig. 2 z-plane

It will also be expressed by the Jacobi's elliptic function as follow:

$$z = \text{sn}(iw). \quad (2)$$

Generalizing the singular points ± 1 and $\pm 1/k$, we put the points $\pm p$ and q in place of them. Then the complex velocity potential w is reduced to the formula (3) by the following transformation.

$$z = z'/q, \quad k = q/p,$$

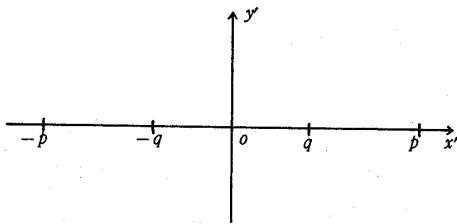


Fig. 3 z'-plane

$$w = -p \int_0^{z'} \frac{dz'}{\sqrt{(p^2 - z'^2)(z'^2 - q^2)}} \quad (3)$$

Then, we transform as shown in Fig.4 the z' -plane into ζ -plane as follows:

$$\zeta = iz', \quad d\zeta = idz'. \quad (4)$$

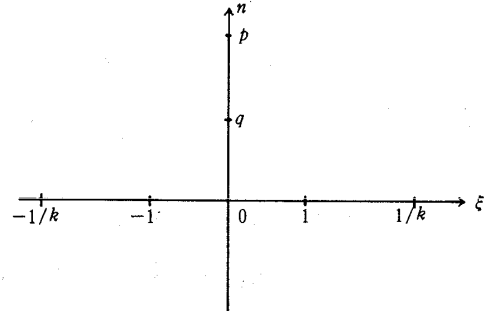


Fig. 4 s-plane

The complex velocity potential w is reduced to the formula (5) as follows:

$$w = -p \int_0^\zeta \frac{d\zeta/i}{\sqrt{(p^2 - \zeta^2/i^2)(\zeta^2/i^2 - q^2)}} \\ = p \int_0^\zeta \frac{d\zeta}{\sqrt{(p^2 + \zeta^2)(q^2 + \zeta^2)}} \quad (5)$$

Still more, if the ζ -plane is transformed into t -plane as shown in Fig. 5, ζ and t are reduced as follows:

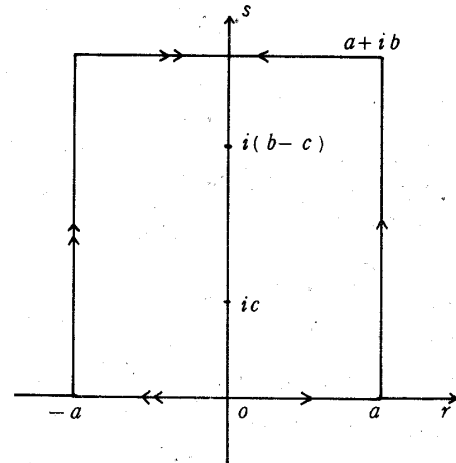


Fig. 5 t-plane

$$\zeta = \text{sn}(t), \quad (6)$$

$$t = \int_0^\zeta \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - k_1^2 \zeta^2)}} = \\ \int_0^1 \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - k_1^2 \zeta^2)}} + \\ \int_1^{1/k_1} \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - k_1^2 \zeta^2)}} +$$

$$\int_{\frac{1}{k_1}}^{\zeta} \frac{d\zeta}{\sqrt{(1-\zeta)(1-k_1^2\zeta)}} = a + ih + \int_{\frac{1}{k_1}}^{\zeta} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k_1^2\zeta^2)}} \dots (7)$$

Transforming the ζ -plane into ζ' -plane by the formula (8) as shown in Fig. 6, we get the formula (9) as follows:

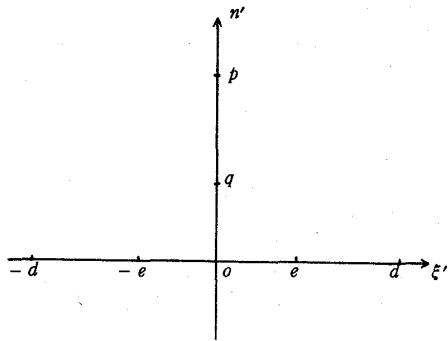


Fig. 6 s' -plane

$$\zeta = \zeta'/e, \quad k_1 = e/d, \quad (8)$$

$$t = -id \int_0^{\zeta'} \frac{d\zeta'}{\sqrt{(d^2 - \zeta'^2)(\zeta'^2 - e^2)}} \quad (9)$$

$$\therefore \zeta = \text{sn}(t) = iz' = iqz,$$

$$z = -\frac{i}{q} \zeta = -\frac{i}{q} \text{sn}(t) = \text{sn}(iw) \quad (10)$$

§ 3 Determination of the values of d, e and p, q.

The values of d and e are determined through solving the following simultaneous equations (11), (12).

$$a = -id \int_0^e \frac{d\zeta'}{\sqrt{(d^2 - \zeta'^2)(\zeta'^2 - e^2)}} \quad (11)$$

$$ib = -id \int_e^d \frac{d\zeta'}{\sqrt{(d^2 - \zeta'^2)(\zeta'^2 - e^2)}} \quad (12)$$

The values of p and q are determined through solving the following equations (13) and (14), in which the value of k, is

calculated from the values of e and d obtained from the preceding formulae (11) and (12).

$$ic = \int_0^{iq} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \quad (13)$$

We get the value of q from this equation (13).

$$i(b-c) = \int_0^{ip} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \quad (14)$$

We get the value of p from this equation (14).

§ 4 Velocity potential and stream function.

Putting as:

$$w = \phi + i\psi, \quad t = r + is,$$

we get the following from the equation (10).

$$\text{sn}(iw) = \text{sn}(-\phi + i\psi) = -\frac{i}{q} \text{sn}(r + is). \quad (16)$$

Through making use of the following addition formulae of Jacobi's elliptic function, we get the formula (18).

$$\left. \begin{aligned} \text{sn}(u+v) &= \frac{\text{sn}u \text{cn}v \text{dn}v + \text{sn}v \text{cn}u \text{dn}u}{1 - k^2 \text{sn}^2u \text{sn}^2v} \\ \text{cn}(u+v) &= \frac{\text{cn}u \text{cn}v - \text{sn}u \text{sn}v \text{dn}u \text{dn}v}{1 - k^2 \text{sn}^2u \text{sn}^2v} \\ \text{dn}(u+v) &= \frac{\text{dn}u \text{dn}v - k^2 \text{sn}u \text{sn}v \text{cn}u \text{cn}v}{1 - k^2 \text{sn}^2u \text{sn}^2v} \end{aligned} \right\} \quad (17)$$

By the preceding formulae, $\text{sn}(u+iv)$ is reduced as follows:

$$\text{sn}(u+iv) = \frac{\text{sn}u \text{cn}iv \text{dn}iv + \text{sn}iv \text{cn}u \text{dn}u}{1 - k^2 \text{sn}^2u \text{sn}^2iv} \quad (18)$$

Through making use of the following imaginary number transformation formulae (19) of the elliptic function, we get the formula (20).

$$\left. \begin{aligned} \operatorname{sn}(iv, k) &= i \frac{\operatorname{sn}(v, k')}{\operatorname{cn}(v, k')} \\ \operatorname{cn}(iv, k) &= \frac{1}{\operatorname{cn}(v, k')} \\ \operatorname{dn}(iv, k) &= \frac{\operatorname{dn}(v, k')}{\operatorname{cn}(v, k')} \end{aligned} \right\} \quad (19)$$

$$\begin{aligned} \operatorname{sn}(u+iv) &= \frac{\frac{\operatorname{sn} \operatorname{dn}(v, k')}{\operatorname{cn}^2(v, k')} + i \frac{\operatorname{sn}(v, k') \operatorname{cn} \operatorname{dn} u}{\operatorname{cn}(v, k')}}{1 + \frac{k^2 \operatorname{sn}^2 u \operatorname{sn}^2(v, k')}{\operatorname{cn}^2(v, k')}} \\ &= \frac{\operatorname{sn} \operatorname{dn}(v, k')}{\operatorname{cn}^2(v, k') + k^2 \operatorname{sn}^2 u \operatorname{sn}^2(v, k')} \\ &\quad + i \frac{\operatorname{sn}(v, k') - \operatorname{cn}(v, k') \operatorname{cn} \operatorname{dn} u}{\operatorname{cn}^2(v, k') + k^2 \operatorname{sn}^2 u} \quad (20) \end{aligned}$$

substituting as follows:

$$u = -\psi, \quad v = \phi$$

we get the following formula (21).

$$\begin{aligned} \operatorname{sn}(-\psi + i\phi) &= \\ &= \frac{\operatorname{sn}(-\psi) \operatorname{dn}(\phi, k')}{\operatorname{cn}^2(\phi, k') + k_1^2 \operatorname{sn}^2(-\psi) \operatorname{sn}^2(\phi, k')} \\ &\quad + i \frac{\operatorname{sn}(\phi, k_1') \operatorname{cn}(\phi, k_1') \operatorname{cn}(-\psi) \operatorname{dn}(-\psi)}{\operatorname{cn}^2(\phi, k_1') + k_1^2 \operatorname{sn}^2(-\psi) \operatorname{sn}^2(\phi, k_1')} \quad (21) \end{aligned}$$

where $\operatorname{sn} u$ is an odd function and both $\operatorname{cn} u$, $\operatorname{dn} u$ are even function as follows:

$$\begin{aligned} \operatorname{sn}(-u) &= -\operatorname{sn}(u) \\ \operatorname{cn}(-u) &= \operatorname{cn}(u) \\ \operatorname{dn}(-u) &= \operatorname{dn}(u) \end{aligned}$$

By making use of the preceding formulae, we can reduce the formula (21) as follows:

$$\begin{aligned} \operatorname{sn}(-\psi + i\phi) &= \\ &= \frac{-\operatorname{sn} \psi \operatorname{dn}(\phi, k_1')}{\operatorname{cn}^2(\phi, k_1') + k_1^2 \operatorname{sn}^2 \psi \operatorname{sn}^2(\phi, k_1')} \\ &\quad + i \frac{\operatorname{sn}(\phi, k_1') \operatorname{cn}(\phi, k_1') \operatorname{cn} \psi \operatorname{dn} \psi}{\operatorname{cn}^2(\phi, k_1') + k_1^2 \operatorname{sn}^2 \psi \operatorname{sn}^2(\phi, k_1')} \end{aligned}$$

Substituting into the formula (17) the following:

$$d = r, \quad v = is,$$

We get the following:

$$\begin{aligned} \operatorname{sn}(r + is) &= \\ &= \frac{\operatorname{sn} r \operatorname{dn}(s, k')}{\operatorname{cn}^2(s, k') + k^2 \operatorname{sn}^2 r \operatorname{sn}^2(s, k')} \\ &\quad + i \frac{\operatorname{sn}(s, k') \operatorname{cn}(s, k') \operatorname{cn} r \operatorname{dn} r}{\operatorname{cn}^2(s, k') + k^2 \operatorname{sn}^2 r \operatorname{sn}^2(s, k')} \quad (23) \end{aligned}$$

Multiplying both sides of the preceding formula (23) with the factor $(-i/q)$ we get the following:

$$\begin{aligned} &= -\frac{i}{q} \frac{\operatorname{sn}(r + is)}{\operatorname{sn}(s, k') \operatorname{cn}(s, k') \operatorname{cn} r \operatorname{dn} r} \\ &= \frac{1}{q \{ \operatorname{cn}^2(s, k') + k^2 \operatorname{sn}^2 r \operatorname{sn}^2(s, k') \}} \\ &\quad + \frac{i}{q} \frac{\operatorname{sn} r \operatorname{dn}(s, k')}{\{ \operatorname{cn}^2(s, k') + k^2 \operatorname{sn}^2 r \operatorname{sn}^2(s, k') \}} \quad (24) \end{aligned}$$

The formula (22) is equal to the formula (24).

Therefore we get the following two formulae (25) and (26), through putting equal the real parts and the imaginary parts of these formulae respectively.

$$\begin{aligned} &= \frac{-\operatorname{sn} \psi \operatorname{dn}(\phi, k_1')}{\operatorname{cn}^2(\phi, k_1') + k_1^2 \operatorname{sn}^2 \psi \operatorname{sn}^2(\phi, k_1')} \\ &= \frac{\operatorname{sn}(s, k') \operatorname{cn}(s, k') \operatorname{cn} r \operatorname{dn} r}{q \{ \operatorname{cn}^2(s, k') + k^2 \operatorname{sn}^2 r \operatorname{sn}^2(s, k') \}} \quad (25) \end{aligned}$$

$$\begin{aligned} &= \frac{\operatorname{sn}(\phi, k_1') \operatorname{cn}(\phi, k_1') \operatorname{cn} \psi \operatorname{dn} \psi}{\operatorname{cn}^2(\phi, k_1') + k_1^2 \operatorname{sn}^2 \psi \operatorname{sn}^2(\phi, k_1')} \\ &= \frac{\operatorname{sn} r \operatorname{dn}(s, k')}{q \{ \operatorname{cn}^2(s, k') + k^2 \operatorname{sn}^2 r \operatorname{sn}^2(s, k') \}} \quad (26) \end{aligned}$$

§5 Conclusion

Numerical values of the velocity potential and the stream function are able to be obtained as the solution of the simultaneous equations of (25) and (26), by means of the graphical method. By making use

of the values of ϕ and ψ obtained from (25) and (26), we are capable of getting the velocity component of the flow by the partial differentiation $u = -\partial\phi/\partial r$, $v = -\partial\phi/\partial s$. The calculation of the numerical values of the velocity potential ϕ and the stream function ψ is craved in the future.

Acknowledgement

The author wishes to confess his heartily gratitude for Prof. Dr. T. Maekawa at the Hiroshima Engineering University for his kind instruction and encomagement.

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